Adverse Selection

- We will solve a procurement problem using a screening mechanism
- Idea: buyer wants to buy from seller, but doesn't know seller's cost

Setup

- Two players, buyer B and seller S
- $v(x)$: value of x units to B
- $c(x, \theta)$: cost of producing x by S depending on his type θ
- Payoffs:

$$u_B(x, t) = v(x) - t$$

$$u_S(x, t, \theta) = t - c(x, \theta)$$

  t is payment from B to S

- Assumptions: $v' > 0, v'' \leq 0, v(0) = 0$
- $c_{x\theta} < 0$ (higher types have lower marginal cost), $c(0, \theta) = 0 \forall \theta$, $c_X > 0$ (positive MC)
- B designs $t(x)$, a nonlinear price schedule specifying a payoff for each quantity
- Given $t(x)$, under some conditions, a seller of type θ will choose a quantity $x(\theta)$ such that marginal cost equals marginal payoff from one more unit:

$$c_x(x(\theta), \theta) = t'(x)$$

- Note: no matter how B designs $t(x)$, lower cost sellers always produce more. We'll show this.
- Easiest to prove using increasing differences
- Note: if there are k (finitely many) types, I only need t to specify payoffs for k product amounts to implement any outcome.

This is an application of the Revelation Principle.

- In equilibrium, given some t, types $\theta_1, \dots, \theta_k$ choose amounts $x_1, \dots, x_k$ respectively, so we can design t_2 that pays $t_2(x_i) = t(x_i)$ and $t_2(x) = 0$ otherwise: t_2 implements the same outcome
- So in the 2 type case, we only need to choose two pairs (x_1, t_1), (x_2, t_2) such that type 1 wants to choose x_1 and 2 chooses x_2
- Another of those reformulations that are mathematically equivalent but make the problem more tractable
- Types θ_1, θ_2: $\Pr(\theta_1) = p, \Pr(\theta_2) = 1 - p$
- Cost functions:

$$c_1(x) \equiv \mathrm{c}(x, \theta_1)$$

$$c_2(x) \equiv \mathrm{c}(x, \theta_2)$$

- B chooses $\{(x_1, t_1), (x_2, t_2)\}$ to solve:

$$\max p(v(x_1) - t_1) + (1 - p)(v(x_2) - t_2)$$

Subject to

$$t_1 - c_1(x_1) \geq 0 \qquad (IR_1)$$

$$t_2 - c_2(x_2) \geq 0 \qquad (IR_2)$$

$$t_1 - c_1(x_1) \geq t_2 - c_1(x_2) \qquad (IC_1)$$

$$t_2 - c_2(x_2) \geq t_1 - c_2(x_1) \qquad (IC_2)$$

- Note: one weird thing about this setup is both types have the same outside option
- Rarely true in reality
- Note 2: the IC conditions are analogous to requiring tangency in the continuous case (but here "tangency" is not meaningful because there are only 2 options)
- Note 3: there may be solutions where we decide to exclude the low type altogether and just offer one pair (x_2, t_2), but we will come back to that later

First best contract

- First best: no adverse selection
- Hence no ICs constraints in principal's problem:

$$\max p(v(x_1) - t_1) + (1-p)(v(x_2) - t_2)$$

Subject to

$$t_1 - c_1(x_1) \geq 0 \qquad (IR_1)$$

$$t_2 - c_2(x_2) \geq 0 \qquad (IR_2)$$

- FB implies IR constraint is binding for both types (otherwise principal could increase profit lowering transfers):

$$t_1 - c_1(x_1) = 0 \qquad (IR_1)$$

$$t_2 - c_2(x_2) = 0 \qquad (IR_2)$$

- Hence $t_1 = c_1(x_1)$ and $t_2 = c_2(x_2)$.

- Substitute back into principal's problem:

$$\max p\big(v(x_1) - c_1(x_1)\big) + (1-p)\big(v(x_2) - c_2(x_2)\big)$$

- We've simplified the problem: from four variables, two constraints to two variables, no constraints.
- Moreover, notice that the problem is separable in x_1 and x_2: one variable doesn't affect the other.
- This will change under adverse selection!
- Solution:

$$p\big(v'(x_1) - c_1'(x_1)\big) = 0 \Longrightarrow v'(x_1) = c_1'(x_1)$$
$$(1-p)\big(v'(x_2) - c_2'(x_2)\big) = 0 \Longrightarrow v'(x_2) = c_2'(x_2)$$

- Marginal benefit equals marginal cost for both types: first-best contract is efficient.

- Here "first-best" means the value that maximizes the total surplus of the principal and agent

- This implies that $x_2 > x_1$ because $c_2' < c_1'$.

Second best contract

- Let's solve the problem under adverse selection, that is, with the IC constraints:

$$\max p(v(x_1) - t_1) + (1-p)(v(x_2) - t_2)$$

Subject to

$$t_1 - c_1(x_1) \geq 0 \qquad (IR_1)$$

$$t_2 - c_2(x_2) \geq 0 \qquad (IR_2)$$

$$t_1 - c_1(x_1) \geq t_2 - c_1(x_2) \qquad (IC_1)$$

$$t_2 - c_2(x_2) \geq t_1 - c_2(x_1) \qquad (IC_2)$$

- Initial question: the IC constraints have a bite? Can't we simply ignore them, and keep the FB contract?
- No...

First result: first-best is not implementable

First-best:

$$\begin{cases} t_1 - c_1(x_1) = 0 \\ t_2 - c_2(x_2) = 0 \end{cases} \Rightarrow \begin{cases} t_1 = c_1(x_1) \\ t_2 = c_2(x_2) \end{cases}$$

Type 2 may choose contract 1 and obtain:

$$u_2(t_1, x_1) = t_1 - c_2(x_1) > t_1 - c_1(x_1) = 0 = t_2 - c_2(x_2) = u_2(t_2, x_2)$$

That is, $u_2(t_1, x_1) > u_2(t_2, x_2)$: type 2 deviates from the first best.

- Intuition: type 2 has an incentive to mimic type 1. This is the central problem for the principal under adverse selection!
- If the principal simply ignores the informational problem, he incurs into two problems:

  1- Efficiency cost: firms always behave as if they were high-cost, either because it's really high-cost, or because it mimics the high-cost one.

  2- Rent to efficient firm: by mimicking the inefficient firm, the efficient one gets utility $t_1 - c_2(x_1) = c_1(x_1) - c_2(x_1)$
- Hence we have to take into account the IC constraints: to avoid mimicking, the principal will need to design contracts to induce the correct self-selection by each type.
- So there will be some distortion with respect to the first-best.
- Intuitively, the principal has two tools to deal with the risk of firm 2 mimicking firm 1:

  - Give firm 2 a higher transfer t_2 to stick to the correct allocation x_2: this is called informational rent.

  - Distort firm 1's contract downwards so as to make it less attractive to firm 2: in the limit, make $x_1 = t_1 = 0$ so that firm 2 has no incentive to mimic firm 1 (it would make zero profit anyway).
- We'll see that the principal will choose both tools: some rent to firm 2, some distortion in the allocation x_1 of firm 1.
- Is this distortion such that one gets $x_1 > x_2$? No, as this would have the opposite effect – in fact, ICs imply that it cannot be the case:

Second result: $x_2 \geq x_1$

Start with the ICs:

$$\begin{cases} t_1 - c_1(x_1) \geq t_2 - c_1(x_2) \\ t_2 - c_2(x_2) \geq t_1 - c_2(x_1) \end{cases}$$

Adding up:

$$-c_1(x_1) - c_2(x_2) \geq -c_1(x_2) - c_2(x_1)$$

$$c_1(x_2) - c_1(x_1) \geq c_2(x_2) - c_2(x_1)$$

$$\int_{x_1}^{x_2} CMg_1(s)ds \geq \int_{x_1}^{x_2} CMg_2(s)ds$$

$$CMg_1 > CMg_2 \Rightarrow x_2 \geq x_1$$

- In the optimal contract, 1's IR constraint will bind but not his IC, and 2's IC constraint will bind but not his IR
- Let's see this through a series of results.

Third result: $IC_2 \text{ and } IR_1 \Rightarrow IR_2 \text{ is slack}$

$$t_2 - c_2(x_2) \underbrace{\geq}_{IC_2} t_1 - c_2(x_1) > t_1 - c_1(x_1) \underbrace{\geq}_{IR_1} 0$$

Hence $t_2 - c_2(x_2) > 0$.

Fourth result: IR_1 is binding

Assume IR_1 not binding

Then Principal might reduce t_1 and t_2 uniformly so that ICs are unaffected

IR_2 would remain active due to the previous result

IR_1 would remain active

Hence all restrictions would be respected with a higher profit for the principal: contradiction.

Fifth result: IC_2 is binding

Assume otherwise.

Principal could reduce t_2 slightly

Type 2 still chooses x_2 and his IR is not violated if change is small enough since it wasn't binding.

1 chooses x_1 even more strongly and his IR is unaffected.

Sixth result: $IC_2\ binding \Rightarrow IC_1\ slack.$

$$t_2 - c_2(x_2) \underbrace{=}_{IC_2 binding} t_1 - c_2(x_1) \Longrightarrow$$

$$t_2 - t_1 = c_2(x_2) - c_2(x_1) =$$

$$\int_{x_1}^{x_2} CMg_2(s)ds < \int_{x_1}^{x_2} CMg_1(s)ds =$$

$$c_1(x_2) - c_1(x_1)$$

Hence:

$$t_2 - t_1 < c_1(x_2) - c_1(x_1)$$

$$t_2 - c_1(x_2) < t_1 - c_1(x_1)$$

That is, IC_1 is slack.

7th result: $t_2 \geq t_1$

$$t_2 - c_2(x_2) \geq t_1 - c_2(x_1)$$

$$t_2 - t_1 \geq c_2(x_2) - c_2(x_1) \geq 0$$

Hence $t_2 - t_1 \geq 0$.

Additionally, $t_2 > t_1$ if $x_2 > x_1$.

- In short: IR_1 and IC_2 binding, other constraints are not binding.
- So B first chooses a point on 1's zero-profit curve, i.e., B chooses x_1 and $t_1 = c_1(x_1)$
- And then moves up 2's cost curve up to some point, i.e., B chooses x_2 and $t_2 = t_1 - c_2(x_1) + c_2(x_2)$
- This is precisely the answer to the mimicking problem (type 2 wants to mimic type 1 in the FB): the principal leaves type 2 exactly indifferent between mimicking or not (that is, IC_2 is binding).
- Problem for the principal: this leads to some 'extra utility' for type 2: IR_2 is not binding: type 2 gets informational rent.
- And how should the principal choose x_1, x_2 ?
- Let's go back to the principal's problem:

$$\max p(v(x_1) - t_1) + (1-p)(v(x_2) - t_2)$$

Now there are only two (binding) constraints: IR_1 and IC_2:

$$t_1 - c_1(x_1) = 0$$

$$t_2 - c_2(x_2) = t_1 - c_2(x_1)$$

We can get t_1 and t_2 from these two conditions:

$$t_1 = c_1(x_1)$$

$$t_2 = c_2(x_2) + t_1 - c_2(x_1) = c_2(x_2) + c_1(x_1) - c_2(x_1)$$

- Notice that now t_2 depends on x_1! It wasn't like this in the first-best.
- This is because x_1 now has two costs for the principal:

  1- the direct production/opportunity cost that must be paid to firm 1: $c_1(x_1)$

  2- the indirect informational rent that must be paid to firm 2 so that it won't mimic firm 1: $c_1(x_1) - c_2(x_1)$

- Notice that $c_1'(x_1) > c_2'(x_1)$ implies that the informational rent $c_1(x_1) - c_2(x_1)$ is increasing in x_1!

$$\frac{d[c_1(x_1) - c_2(x_1)]}{dx_1} = c_1'(x_1) - c_2'(x_1) > 0$$

- Hence by decreasing x_1, the principal may decrease the informational rent to firm 2.

Summay

- IR_1 and IC_2 are binding
- IR_2 and IC_1 are not binding and, moreover, are implied by IC_2 and IR_1, so we can effectively ignore them

Principal's problem

- Let's plug these conditions into the principal's objective function:

$$\max p\left(v(x_1) - \underbrace{c_1(x_1)}_{t_1}\right) + (1-p)\left(v(x_2) - \left(\underbrace{c_2(x_2) + c_1(x_1) - c_2(x_1)}_{t_2}\right)\right)$$

- The terms in red are the difference with respect to the first-best problem.
- Solution:

$$p\big(v'(x_1) - c_1'(x_1)\big) - (1-p)\big(c_1'(x_1) - c_2'(x_1)\big) = 0 \Longrightarrow$$

$$v'(x_1) - c_1'(x_1) = \frac{(1-p)}{p}\underbrace{\big(c_1'(x_1) - c_2'(x_1)\big)}_{>0} > 0$$

$$(1-p)\big(v'(x_2) - c_2'(x_2)\big) = 0 \Longrightarrow v'(x_2) = c_2'(x_2)$$

- x_2 can just be picked as first-best!
- Whatever x_1 is, changing x_2 does not affect 1 's incentives, just how much 2 produces and how much B pays 2
- So can just choose x_2 such that $c_2'(x_2) = v'(x_2)$ (first-best):

$$x_2^* = x_2^{FB}$$

- However, $v'(x_1) > c_1'(x_1)$ and x_1 is lower then in the first-best:

$$x_1^* < x_1^{FB}$$

- Picking the first-best x_1 is not good: the more I increase x_1, not only do I have to pay 1 more, but also have to pay 2 more at the same x_2 to satisfy his IC: this is an additional cost due specifically to the adverse selection problem.
- For the same reason, x_1 higher than FB is also bad, and optimal x_1 is below FB
- If $p < c_1'(x_1) - (1-p)c_2'(x_1)$ even for small x_1, then may want to choose $x_1 = 0$ (type 1 out of the market)

- p does not affect x_2, but it affects x_1
- The lower p is, the lower x_1 is
- Main tension in this model is between desire to produce at the efficient level (choose x_1, x_2 equal to FB levels) and B's desire to limit type 2's rent
- Two tools to reduce type 2's rent: give type 2 informational rent (IR_2 not binding) and distort downwards the allocation of type 1 (x_1 below first-best level)
- Have to screw over type 1 to reduce type 2's temptation!
- If p is low, lowering x_1 has low efficiency cost (low type is unlikely anyway) but big rent reduction (B pays less to the likely high type)
- Vice versa for high p
- Remark: we hid one additional constraint while solving the problem:

$$x_2 \geq x_1$$

- But the condition $x_2 \geq x_1$ does not bind so we can ignore it (we'll come back to this in the continuous case)
- And also the value that would result from the optimal contract if the agent were known to be type 2
- From the first FOC,

$$v'(x_1) - c_1'(x_1) = \frac{1-p}{p}\left(c_1'(x_1) - c_2'(x_1)\right) > 0$$

- so $x_1^* < x_1^{FB}$
- Hence the principal designs the menu $\{(x_1, t_1), (x_2, t_2)\}$ so that type 1 underproduces in equilibrium
- Again, this is to make it cheaper to prevent type 2's temptation to fake being type 1
- In particular, $x_2^* = x_2^{FB} > x_1^{FB} > x_1^*$
- If p is high, there is less distortion in x_1 so x_1^* goes up
- If p is low enough, can go all the way to $x_1 = 0$ (type 1 is shut out of the market)

Virtual Cost Function

- An alternative way to think about the problem of choosing x_1
- We can define

$$\tilde{c}(x_1) \equiv c_1(x_1) + \frac{1-p}{p} \Delta c(x_1)$$

- Then the choice of x_1 made in the screening mechanism is actually the FB choice, for a hypothetical agent that had this (higher) cost function
- The cost function captures both the real cost of 1 producing more x, and the cost of having to pay type 2 more as a result of increasing x_1

m-state case

- Suppose I have types $\theta_1, \ldots, \theta_m$
- Cost functions $c_1, \ldots, c_m$ such that $c_i'(x) > c_j'(x)$ for all $i < j$ and any x (higher types have lower marginal cost)
- Probabilities $p_1, \ldots, p_m$ adding up to 1
- How to design the mechanism?
- As before, we need to define at most m points: $(t_1, x_1), \ldots, (t_m, x_m)$
- Could be fewer if I want to shut out some types, but not more (can just drop options from the contract which no one picks in equilibrium anyway)
- Now there are m IR constraints: $\text{IR}_1, \ldots, \text{IR}_m$
- How many IC constraints? For each type k, need one IC constraint for each $i \neq k$, saying k prefers picking (t_k, x_k) to (t_i, x_i)
- So $k(k-1)$ IC constraints: $\text{IC}_{k1}, \ldots, \text{ICI}_{k(k-1)}, \text{IC}_{k(k+1)}, \ldots, \text{ICI}_{kn}$
- Which ones bind?
- We can show (with similar arguments to the 2-state case) that:
- Only IR_1 binds (higher types have lower cost so necessarily positive profits)

- Only $\mathrm{IC}_{k(k-1)}$ binds for each $k = 2, \dots, n$
- Lowest type who is not priced out is left indifferent about entering
- Each type is indifferent about not mimicking the next type with higher cost
- (But strictly does not want to mimic others)
- This gives the right amount of conditions: given some values of $x_1, \dots, x_m$, the conditions uniquely pin down $t_1, \dots, t_m$
- From IR_1, we know $t_1 = c_1(x_1)$: pins down t_1
- From IC_{21}, we know that $t_2 - c_2(x_2) = t_1 - c_2(x_1)$: pins down t_2
- And so on
- Finding the optimal $x_1, \dots, x_m$ still requires solving for some FOCs
- (Side note: choosing t_i 's with this algorithm allows us to implement any sequence $x_1, \dots, x_m$ we want, as long as it's non-decreasing, but some are better for the principal than others)
- $x_m^* = x_m^{FB}$, but for $i < m$ we will have $x_i^* < x_i^{FB}$
- As before, increasing x for low types forces principal to pay all higher types more (by the same amount)
- Hence distortion is worst for the lowest i's (highest cost types)

Continuous Case

- Suppose now we have a continuum of types $\theta \in [\underline{\theta}, \overline{\theta}]$
- θ distributed according to a continuous cdf F, with density f
- (Could deal with atoms in distribution; holes in the support are more annoying)
- Suppose $c_{x\theta} < 0, c(0, \theta) = 0$ for all θ, and (hence) $c_\theta < 0$
- Higher types have lower marginal cost, hence lower cost

Now principal solves:

$$\max_{x(\cdot),t(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} (v(x(\theta)) - t(\theta))f(\theta)d\theta$$
$$\text{s.t. } t(\theta) - c(x(\theta),\theta) \geq t(\theta') - c(x(\theta'),\theta) \ \ \forall\theta,\theta'$$
$$t(\theta) - c(x(\theta),\theta) \geq 0 \ \ \forall\theta$$

- Define $\Pi(\tilde{\theta},\theta) \equiv t(\tilde{\theta}) - c(x(\tilde{\theta}),\theta))$
- This is the profit θ gets from pretending to be $\tilde{\theta}$
- Define $V(\theta) \equiv \Pi(\theta,\theta)$
- This is type θ 's equilibrium payoff: that is, when type θ truthfully reports $\tilde{\theta} = \theta$
- Remember that due to adverse selection, type θ can pretend (report) to be any other type $\tilde{\theta} \neq \theta$
- So we're interested in knowing what type θ gets when he reports $\tilde{\theta} = \theta$ (the Revelation Principle implies that we can focus on equilibria with truthful reporting – we'll get back to this when discussing mechanism design)
- Then the IC conditions can be rewritten as $V(\theta) \geq \Pi(\tilde{\theta},\theta)$ for all $\theta, \tilde{\theta}$
- What do our conditions imply about $V(\theta)$?
- Since it's the value function of an optimization problem, we can use the envelope theorem:

$$V'(\theta) = \frac{d\Pi(\theta,\theta)}{d\theta} = \left.\frac{\partial\Pi(\theta_1,\theta_2)}{\partial\theta_2}\right|_{(\theta,\theta)} = -c_\theta(x(\theta),\theta) > 0$$

- Note: $V(\theta)$ a priori doesn't have to be differentiable, as it is endogenous: the principal could pick a non-smooth x or t
- But we know c_θ is well-defined by assumption
- There are versions of the envelope theorem for non-differentiable functions, which guarantee we can use it without knowing ex ante that V is differentiable
- But too complicated for this class, so just assume functions are differentiable
- "But I don't remember the envelope theorem". No problem, just write:

$$V(\theta) \equiv \Pi(\theta,\theta) = t(\theta) - c(x(\theta),\theta))$$

Hence:

$$V'(\theta) = \frac{d\Pi(\theta,\theta)}{d\theta} = \underbrace{t_\theta(\theta) - c_x(x(\theta),\theta)) \cdot x_\theta(\theta)}_{=0} - c_\theta(x(\theta),\theta)) = -c_\theta(x(\theta),\theta))$$

- Why $t_\theta(\theta) - c_x(x(\theta),\theta)) \cdot x_\theta(\theta) = 0$?
- Because the agent chooses his reported type $\tilde{\theta}$ to maximize his utility.
- The agent's problem is: choose $\tilde{\theta}$ to maximize $\Pi(\tilde{\theta},\theta) \equiv t(\tilde{\theta}) - c(x(\tilde{\theta}),\theta))$
- The FOC is exactly $t_\theta(\theta) - c_x(x(\theta),\theta)) \cdot x_\theta(\theta) = 0$
- So we have:

$$\frac{dV(\theta)}{d\theta} = -c_\theta(x(\theta),\theta))$$

- Hence:

$$dV(\theta) = -c_\theta(x(\theta),\theta))d\theta$$

- Integrate both sides, and remember to change the notation for the variable of integration because we'll use θ as a limit of integration:

$$\int_{\underline{\theta}}^{\theta} dV(s) = \int_{\underline{\theta}}^{\theta} -c_\theta(x(s),s))ds$$

$$V(\theta) - V(\underline{\theta}) = \int_{\underline{\theta}}^{\theta} -c_\theta(x(s),s))ds$$

- Now we can use the fact that $V(\underline{\theta}) = \Pi(\underline{\theta},\underline{\theta})$ is the equilibrium utility of the lowest type, and the principal can make it equal to the outside option, which we set at zero.
- So: $V(\underline{\theta}) = 0$
- This is the same argument we made in the two-type case to conclude that IR_1 was binding.

- We get:

$$V(\theta) = \int_{\underline{\theta}}^{\theta} -c_\theta(x(s), s))ds$$

- But we also have:

$$V(\theta) = t(\theta) - c(x(\theta), \theta)$$

- Hence:

$$t(\theta) - c(x(\theta), \theta) = \int_{\underline{\theta}}^{\theta} -c_\theta(x(s), s))ds$$

$$t(\theta) = c(x(\theta), \theta) + \int_{\underline{\theta}}^{\theta} -c_\theta(x(s), s))ds$$

- That is, we have an expression for $t(\theta)$, as we had in the discrete case.
- Again, it has two parts:

  1- Direct production/opportunity cost: $c(x(\theta), \theta)$

  2- Informational rent: $\int_{\underline{\theta}}^{\theta} -c_\theta(x(s), s))ds$

Digression

- This has a similar flavor to the finite types case: given some $x(\theta)$, we can pin down $t(\theta)$
- But it is not logically equivalent!
- In the finite case, given $x_1, \dots, x_m$, there were many $t_1, \dots, t_m$ that could be paired with them that would implement production x_1 for $\theta_1, \dots, x_m$ for θ_m
- The uniqueness of the t_i followed from making some IR and IC conditions bind, to achieve optimality for the principal
- (You could design other t_i schedules such that no IR or ICs would bind, and which would also implement the same x_i 's, but they would give some agent types free money)

- On the other hand, in the continuous case, the conditions which uniquely pin down $V(\theta)$ and $t(\theta)$ (up to $\Pi(0,0)$) follow exclusively from the assumption that picking the schedule $x(\theta)$ is optimal (i.e., incentive compatible) for the agent
- We have not yet exploited in any way the assumption that we're trying to achieve optimality for the principal!
- The only way optimality for the principal will show up, in terms of conditions on t, is that we should set $\Pi(0,0) = 0$ (no free money for lowest type)
- End of digression!

New version of principal's problem

- We still have to find the optimal schedule $x(\theta)$
- The original problem is:

$$\max_{x(\cdot),t(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} (v(x(\theta)) - t(\theta))f(\theta)d\theta$$
$$\text{s.t. } t(\theta) - c(x(\theta),\theta) \geq t(\theta') - c(x(\theta'),\theta) \;\; \forall\theta,\theta'$$
$$t(\theta) - c(x(\theta),\theta) \geq 0 \;\; \forall\theta$$

- Using the expression we found for $t(\theta)$, we may rewrite it as:

$$\max_{x(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} \left[v(x(\theta)) - \underbrace{\left[c(x(\theta),\theta) + \int_{\underline{\theta}}^{\theta} -c_\theta(x(s),s)ds \right]}_{t(\theta)} \right] f(\theta)d\theta$$

- Subject only to the condition that $x(\theta)$ is non-decreasing

New version of principal's problem

- We have a double integral in the principal's problem. Write it as follows:

$$\max_{x(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} [v(x(\theta)) - c(x(\theta),\theta)]f(\theta)d\theta + \int_{\underline{\theta}}^{\overline{\theta}} \left(\int_{\underline{\theta}}^{\theta} c_\theta(x(s),s))ds \right) f(\theta)d\theta$$

- Let's work on the double integral. We'll use integration by parts, defining:

$$\int_{\underline{\theta}}^{\overline{\theta}} \underbrace{\left(\int_{\underline{\theta}}^{\theta} c_\theta(x(s),s))ds \right)}_{u} \underbrace{f(\theta)}_{v'} d\theta$$

- Remember integration by parts:

$$\int_{\underline{\theta}}^{\overline{\theta}} u(\theta) \cdot v'(\theta)d\theta = \left[u(\overline{\theta}) \cdot v(\overline{\theta}) - u(\underline{\theta}) \cdot v(\underline{\theta})\right] - \int_{\underline{\theta}}^{\overline{\theta}} u'(\theta) \cdot v(\theta)d\theta$$

- In our case:

$$u(\theta) = \int_{\underline{\theta}}^{\theta} c_\theta(x(s),s)ds \Longrightarrow u'(\theta) = c_\theta(x(\theta),\theta)$$

$$v'(\theta) = f(\theta) \Longrightarrow v(\theta) = F(\theta)$$

- To compute $u'(\theta)$, we used Leibnitz's rule:

$$\frac{d}{d\theta}\left(\int_{b(\theta)}^{a(\theta)} c(\theta,s)ds\right) = a'(\theta)c(\theta,\theta) - b'(\theta)c(\theta,\theta) + \int_{b(\theta)}^{a(\theta)} \frac{d}{d\theta} c(\theta,s)ds.$$

  In our case:

  $a(\theta) = \theta$, hence $a'(\theta) = 1$

  $b(\theta) = \underline{\theta}$, hence $b'(\theta) = 0$

  $c(\theta,s) = c_\theta(x(s),s))$, hence $\frac{d}{d\theta}c(\theta,s) = 0$

- Hence:

$$\int_{\underline{\theta}}^{\overline{\theta}} \underbrace{\left(\int_{\underline{\theta}}^{\theta} c_\theta(x(s),s))ds\right)}_{u} \underbrace{f(\theta)}_{v'}\, d\theta$$

$$= \left[\left(\int_{\underline{\theta}}^{\overline{\theta}} c_\theta(x(\theta),\theta))d\theta\right) \underbrace{F(\overline{\theta})}_{1} - \underbrace{\left(\int_{\underline{\theta}}^{\underline{\theta}} c_\theta(x(\theta),\theta))d\theta\right) F(\underline{\theta})}_{=0}\right]$$

$$- \int_{\underline{\theta}}^{\overline{\theta}} c_\theta(x(\theta),\theta) \cdot F(\theta)d\theta$$

$$= \int_{\underline{\theta}}^{\overline{\theta}} c_\theta(x(\theta),\theta))d\theta - \int_{\underline{\theta}}^{\overline{\theta}} c_\theta(x(\theta),\theta) \cdot F(\theta)d\theta =$$

$$\int_{\underline{\theta}}^{\overline{\theta}} c_\theta(x(\theta),\theta)) \cdot \big(1 - F(\theta)\big)d\theta =$$

$$\int_{\underline{\theta}}^{\overline{\theta}} c_\theta(x(\theta),\theta)) \cdot \frac{\big(1 - F(\theta)\big)}{f(\theta)} f(\theta)d\theta$$

- In short, we have shown that:

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(\int_{\underline{\theta}}^{\theta} c_\theta(x(s),s))ds\right) f(\theta)d\theta = \int_{\underline{\theta}}^{\overline{\theta}} c_\theta(x(\theta),\theta)) \cdot \frac{\big(1 - F(\theta)\big)}{f(\theta)} f(\theta)d\theta$$

- Hence we can rewrite the principal's problem as:

$$\max_{x(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} [v(x(\theta)) - c(x(\theta),\theta)]f(\theta)d\theta + \int_{\underline{\theta}}^{\overline{\theta}} c_\theta(x(\theta),\theta)) \cdot \frac{\big(1 - F(\theta)\big)}{f(\theta)} f(\theta)d\theta$$

$$\max_{x(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} \left[v(x(\theta)) - c(x(\theta),\theta) + c_\theta(x(\theta),\theta)) \cdot \frac{\big(1 - F(\theta)\big)}{f(\theta)}\right] f(\theta)d\theta$$

Or:

$$\max_{x(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} [v(x(\theta)) - \tilde{c}(x(\theta),\theta)]f(\theta)d\theta$$

where

$$\tilde{c}(x(\theta), \theta) \equiv c(x, \theta) - c_\theta(x, \theta)\left(\frac{1 - F(\theta)}{f(\theta)}\right)$$

- That is, total cost for a given $x(\theta)$ has two parts:

  1- Production cost (including opportunity cost): $c(x, \theta)$

  2- Informational rent: $-c_\theta(x, \theta)\left(\frac{1-F(\theta)}{f(\theta)}\right)$, which is increasing in θ

- Deriving with respect to each $x(\theta)$, we get the FOC:

$$v'(x(\theta)) = c_x(x, \theta) - c_{x\theta}(x, \theta)\left(\frac{1 - F(\theta)}{f(\theta)}\right) \quad \forall\theta$$

- This gives us an equation in $x(\theta)$ which generally pins down $x(\theta)$
- As before, the solution satisfies that $x^*(\theta) < x^{FB}(\theta)$ for $\theta < \overline{\theta}$, and $x^*(\overline{\theta}) = x^{FB}(\overline{\theta})$
- One question left: is the solution $x^*(\theta)$ pinned down by this condition necessarily non-decreasing?
- Not always!
- It turns out that, when the solution to this system of FOCs is non-monotonic, you can find the "real" solution by smoothing out the decreasing parts
- Surprisingly, this does not affect the optimal value of $x(\theta)$ outside of the regions we're smoothing out
- This is because of the agent's quasilinear utilities: changing x, and t, for some θ affects required payoffs for all θ 's equally, so does not affect local incentives