Repeated Games

Consider a static game $\Gamma_N = [I, \{S_i\}_i, \{u_i\}_i]$. In a repeated game, players play this game in periods $t = 1,2, \dots, T$, with either $T < \infty$ or $T = \infty$.

We have the following results:

1. Playing the static NE in every period is a SPNE.
2. If Γ_N has only one Nash Equilibrium and $T < \infty$, then the only SPNE of the repeated game is the repetition of the static equilibrium in every period.

We interpret this as the impossibility of cooperation (ie, coordinating on something different from the NE).

Proof is simple and follow from the impossibility of credible rewards (for cooperation) or punishments (for non-cooperation).

Games with multiple Nash equilibria

If the static game has multiple equilibria, it may be possible to build different SPNE.

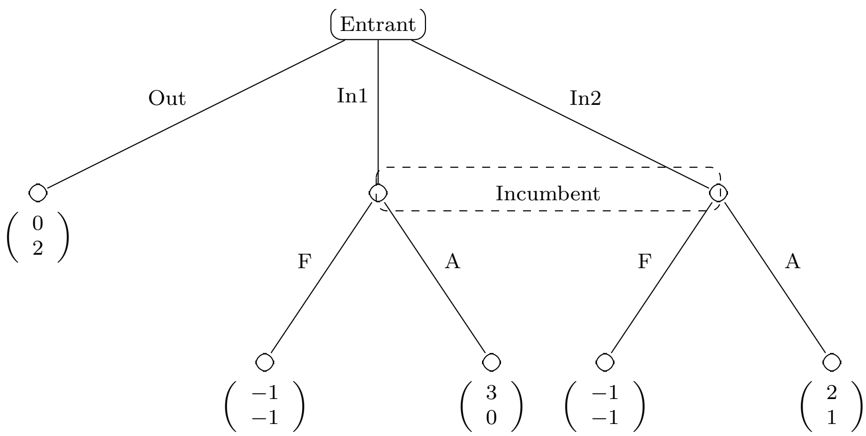
Example:

	A	B	C	D
A	3,3	1,1	1,1	1,1
B	1,1	7,7	1,8	1,1
C	1,1	8,1	1,1	1,1
D	1,1	1,1	1,1	5,5

Assume this game is played twice.

SPNE: play B in the first period; if (B, B) was the outcome of the first period, play D in the second period; otherwise play A.

Beliefs and Sequential Rationality



	F	A
Out	0,2	0,2
In1	-1,-1	3,0
In2	-1,-1	2,1

Problem: the only subgame is the whole game. Hence all NE are SPNE: $\{(Out, F), (In_1, A)\}$.

We cannot use subgame perfection to rule out the non-credible threat that leads to (Out, F).

It is non-credible because the Incumbent prefers A to F for any belief $[\mu, (1-\mu)]$ that it may have over its decision nodes:

$$u_I^e(F|\mu) = \mu \cdot (-1) + (1-\mu) \cdot (-1) = -1$$

$$u_I^e(A|\mu) = \mu \cdot (0) + (1-\mu) \cdot (1) = 1 - \mu > -1 \ \forall \mu \in [0,1]$$

Hence $\forall \mu$, $A \succ_I F$: F is non-credible.

The Incumbent's strategy must be optimal for some belief μ about the Entrant's choice.

We need to extend the idea of sequential rationality to include beliefs:

We want to apply it to parts of the game that look like a subgame but do not begin in an individual decision node.

Definition: **System of Beliefs**: $\mu \in \Gamma_E: \mu(x) \in [0,1]$ for all decision node x such that $\sum_{x\in H} \mu(x) = 1$, for all information set H.

It is simply a distribution over nodes for each information set.

That is, the player gives a probability for previous plays by other players, conditional to a given information set being reached.

Now we can define sequential rationality:

Definition: strategy profile $\sigma = (\sigma_1, \dots, \sigma_I)$ in Γ_E is sequentially rational given μ if:

$$E\left[u_{i(H)}|H, \mu, \sigma_{i(H)}, \sigma_{-i(H)}\right] \geq \left[u_{i(H)}|H, \mu, \tilde{\sigma}_{i(H)}, \sigma_{-i(H)}\right]$$

For all $\tilde{\sigma}_{i(H)} \in \Delta\left(S_{i(H)}\right)$.

That is, expected utility depends now on two sources of uncertainty: other players' actions in H ($\sigma_{-i(H)}$), and previous actions (μ).
If this condition holds for every information set H, then σ is sequentially rational given μ.

That is, no player would like to change his strategy when information set H is reached, given μ and σ_i.

Now we may define a (Weak) Perfect Bayesian Equilibrium:

i- Strategies are sequentially rational.
ii- Beliefs are consistent with strategies whenever possible.

The second point means that beliefs must be correct in equilibrium.

Consistent Beliefs

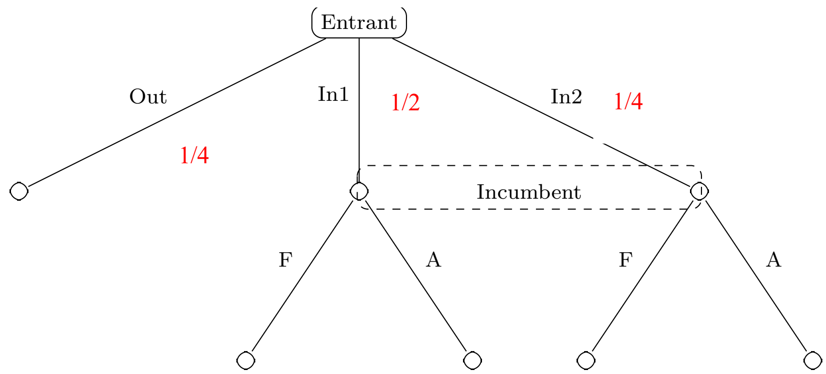
Consider initially only completely mixed strategies: there is a strictly positive probability for every action, in every information set.

Then every information set is reached is reached with a strictly positive probability.

We may now use Bayes' Theorem:

$$Prob(x|H,\sigma) = \frac{Prob(x|\sigma)}{\sum_{x'\in H} Prob(x'|\sigma)}$$

Example: consistent beliefs using Bayes



$$\sigma_E = \left({}^1/_4, {}^1/_2, {}^1/_4\right)$$

$$Prob(In_1 \ or \ In_2) = \frac{1}{4} + \frac{1}{2} = \frac{1}{2}$$

Hence:

$$\mu = Prob(In_1 | In_1 \ or \ In_2) = \frac{{}^1/_2}{{}^3/_4} = \frac{2}{3}$$

This is the incumbent's belief μ consistent with the entrant's strategy σ.

If the probability of a given information set is zero, that we may choose any distribution μ for nodes in this information set: we cannot use Bayes.

We have a formal definition of equilibrium now:

Definition: (σ, μ) is a Weak Perfect Bayesian Equilibrium in Γ_E if:

i- σ is sequentially rational given μ.

ii- If $Prob(H|\sigma) > 0$, then $\mu(x) = \frac{Prob(x|\sigma)}{Prob(H|\sigma)}$ for all $\forall x \in H$.

Importantly, the equilibrium concept takes into account both strategies **and beliefs**.

Beliefs are as important as strategies!

Proposition: σ is a Nash Equilibrium in Γ_E if and only if there is μ such that:

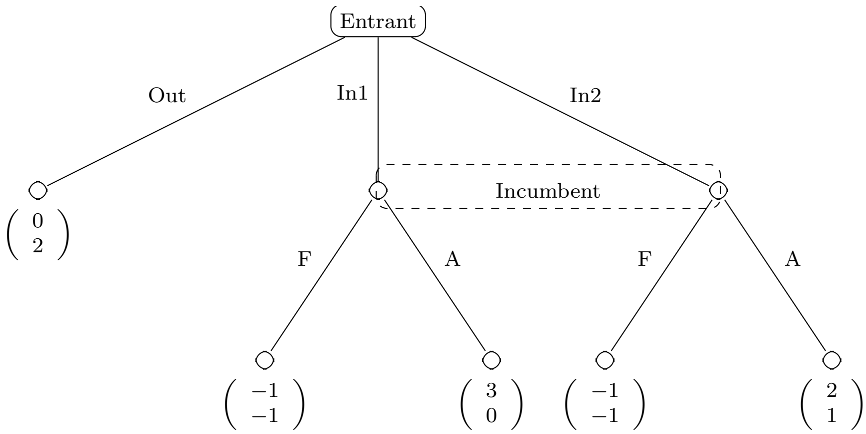
i- σ is sequentially rational given μ, $\forall H \mid Prob(H|\sigma) > 0$.

ii- μ is obtained through Bayes' rule whenever possible.

Hence a Nash Equilibrium does not impose rationality is information sets such that $Prob(H|\sigma) = 0$.

In a Bayesian Perfect Equilibrium, we add sequential rationality for all information sets, even those out of the equilibrium path. (But we don't impose consistency in those sets.)

Example:



Remember we have two NE's / SPNE's: $\{(Out, F), (In_1, A)\}$. But (Out, F) is off the equilibrium path.-

But $u_I^e(F|\mu) < u_I^e(A|\mu)$ for all $\mu \in [0,1]$: there is no belief that makes it optimal for the Incumbent to fight after entry.

(Out, F) is a non-credible threat.

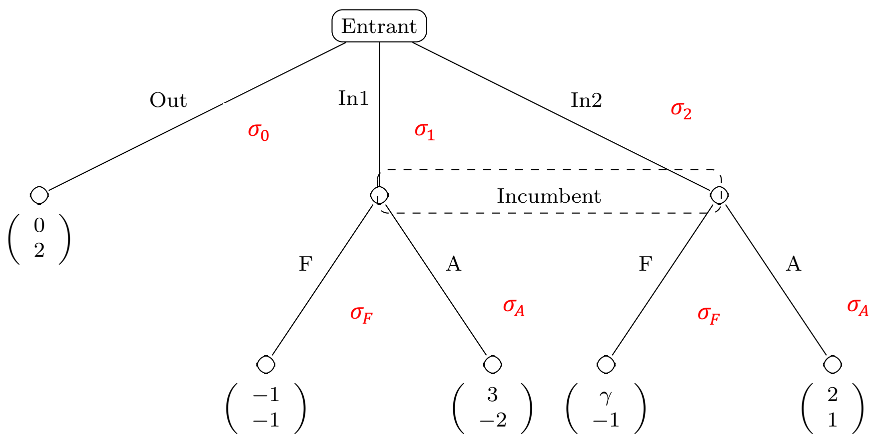
Then (Out, F) cannot be part of wBPE because there is no belief system μ such that F is optimal for the incumbent, once his information set is reached.

To show that (In_1, A) is a wBPE, we need to find μ:

$$\mu = Prob(In_1 | In_1 \ or \ In_2) = \frac{Prob(In_1|In_1)}{Prob(In_1|In_1) + Prob(In_2|In_1)} = \frac{1}{1+0}$$

wBPE: $\{In_1, (A, \mu = 1)\}$.

Example 9.C.3: Compute wPBE with (non-trivial) mixed strategies



(Assume $\gamma > 0$)

Incumbent plays Fight with strictly positive probability iff:

$$u_I^e(F) = \mu \cdot (-1) + (1-\mu) \cdot (-1) = -1 \geq \mu \cdot (-2) + (1-\mu) \cdot 1 = u_I^e(A)$$

$$-1 \geq -2\mu + 1 - \mu$$

$$-1 \geq 1 - 3\mu$$

$$3\mu \geq 2$$

$$\mu \geq {}^2/_3$$

If $\mu > {}^2/_3$, then Incumbent chooses Fight since $F \succ_I A$.

But then the Entrant chooses In_2: $\sigma_0 = \sigma_1 = 0$, $\sigma_2 = 1$.

Bayes then implies $\mu = \frac{0}{0+1} = 0$: absurd because we assumed $\mu > {}^2/_3$.

Strategies and beliefs are not compatible.

Hence one cannot have $\mu > {}^2/_3$.

If $\mu < {}^2/_3$, then Incumbent chooses Accommodate since $A \succ_I F$.

But then the Entrant chooses In_1: $\sigma_0 = \sigma_2 = 0$, $\sigma_1 = 1$.

Bayes then implies $\mu = \frac{1}{0+1} = 1$: absurd because we assumed $\mu < {}^2/_3$.

Again, strategies and beliefs are non compatible.

It follows that in any wPBE, $\mu = {}^{2}/_{3}$, so that $1 - \mu = {}^{1}/_{3}$. That is, $\mu = 2 \cdot (1 - \mu)$.

This implies $\sigma_1 = 2 \cdot \sigma_2$.

And this implies $\sigma_1, \sigma_2 \in (0,1)$: otherwise Bayes would imply either $\mu = 0$ or $\mu = 1$, leading again to an inconsistency.

It follows that in equilibrium, the Entrant must be indifferent between In_1 and In_2.

(We saw that a player must be indifferent between pure strategies in the support of a mixed strategy.)

$$u_E(In_1) = \sigma_F \cdot (-1) + (1 - \sigma_F) \cdot 3 = \sigma_F \cdot (\gamma) + (1 - \sigma_F) \cdot 2 = u_E(In_1)$$

$$\sigma_F + 3 - 3 \cdot \sigma_F = \gamma \cdot \sigma_F + 2 - 2 \cdot \sigma_F$$

$$-4 \cdot \sigma_F + 2 \cdot \sigma_F - \gamma \cdot \sigma_F = -1$$

$$\sigma_F \cdot (1 + \gamma) = 1$$

$$\sigma_F = \frac{1}{\gamma + 2}, \sigma_A = \frac{\gamma + 1}{\gamma + 2}$$

Now we may compute the Entrant's payoffs:

$$u_E^e(In_1) = \frac{1}{\gamma + 2} \cdot (-1) + \frac{\gamma + 1}{\gamma + 2} \cdot 3 = \frac{-1 + 3 \cdot \gamma + 3}{\gamma + 2} = \frac{3\gamma + 2}{\gamma + 2} > 0 = u_E^e(Out)$$

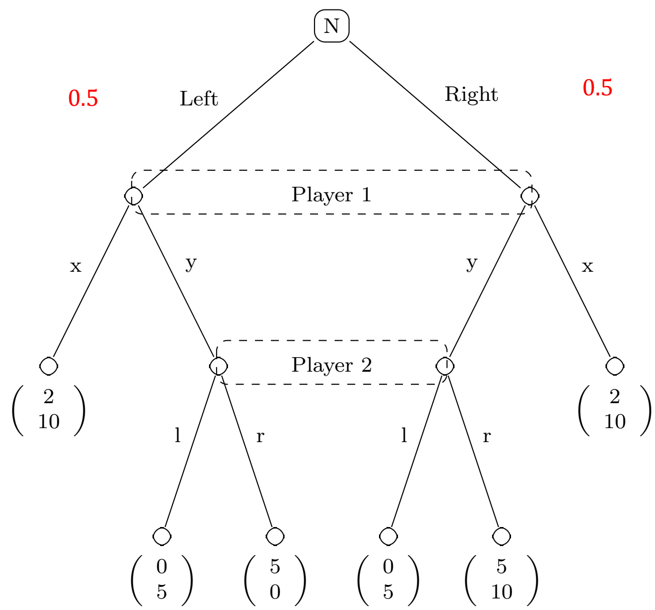
Hence $u_E^e(In_1) = u_E^e(In_2) > u_E^e(Out)$, and $\sigma_0 = 0$.

Then $\sigma_1 + \sigma_2 = 1$. But we saw that $\sigma_1 = 2 \cdot \sigma_2$. Hence $(\sigma_0, \sigma_1, \sigma_2) = \left(0, {}^{2}/_{3}, {}^{1}/_{3}\right)$.

This concludes our example: we have strategies and beliefs, and they are compatible.

wBPE may be too weak: no restrictions on beliefs off the equilibrium path.

Example 9.C.4:



We may build the following: wPBE: $\{[x, (0.5,0.5)], [l, (0.9,0.1)]\}$

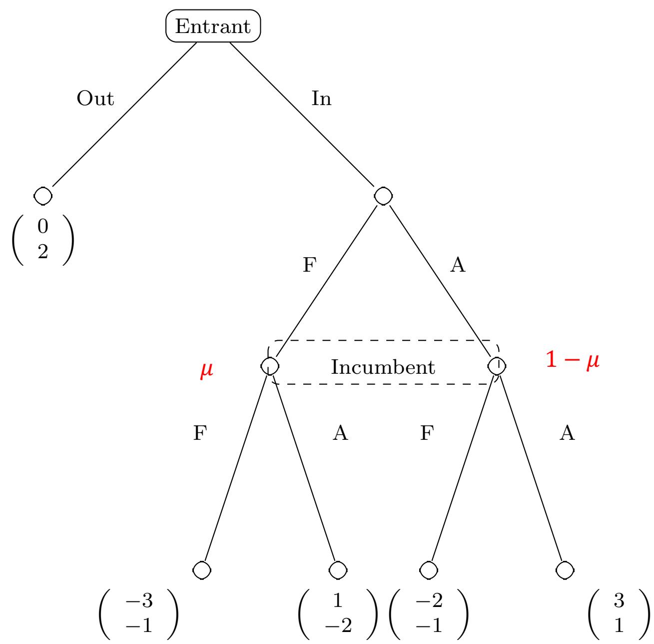
In this equilibrium, player 2 information set is never reached, so we may choose any beliefs.

We chose beliefs $(0.9,0.1)$, leading to $l \succ_2 r$, leading to $x \succ_1 y$.

But $(0.9,0.1)$ doesn't make much sense because Nature uses $(0.5,0.5)$: with these beliefs for player 2, $r \succ_2 l$ and then $y \succ_1 x$.

Leaving beliefs unrestricted in some part of the game is as bad as leaving actions unrestricted!

Example 9.C.5: yet another problem for the wPBE



One wPBE: $\{(Out, A), (F, \mu = 1)\}$.

But only (A, A) is NE in the subgame. (And one may build another wPBE.)

The problem is that a Weak Perfect Bayesian Equilibrium does not need to be Subgame Perfect.

Then we define Perfect Bayesian Equilibrium: it is a wPBE that induces a wPBE in every subgame.

Same reasoning as in SPNE, which induces a NE in every subgame.

An alternative is to consider a Sequential Equilibrium:

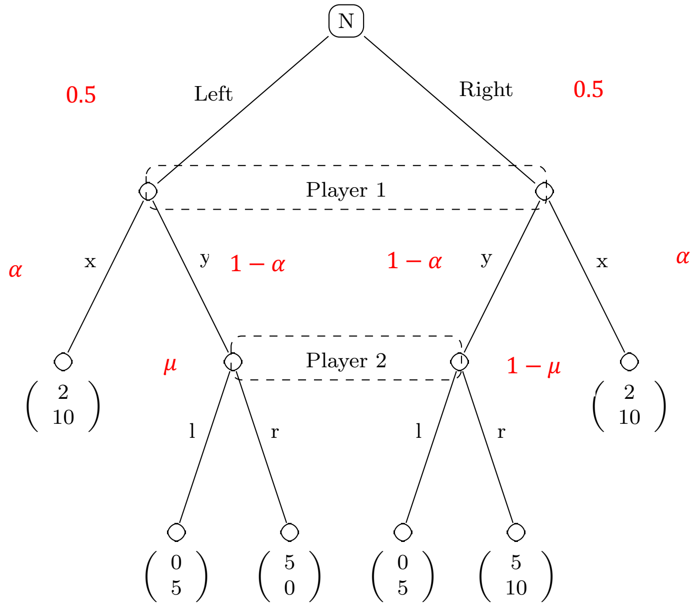
Definition: (σ, μ) is a Sequential Equilibrium in Γ_E if:

i- σ is sequentially rational given μ.

ii- There is a sequence $\{\sigma_k\}$ of completely mixed strategies with $\lim \sigma_k = \sigma$ such that $\mu = \lim \mu_k$, in which μ_k is derived from σ_k using Bayes' rule.

Every sequential equilibrium is a wBPE, but it does not hold in the opposite direction.

Example: back to 9.C.4



$(\alpha, 1-\alpha)$ are mixed strategies.

Use Bayes:

$$\mu(\alpha) = \frac{Prob(left\ node)}{Prob(left\ node) + Prob(right\ node)} = \frac{0.5 \cdot (1-\alpha)}{0.5 \cdot (1-\alpha) + 0.5 \cdot (1-\alpha)} = 0.5$$

Then: $\lim \mu(\alpha) = \lim \frac{1}{2} = \frac{1}{2}$.

In any sequential equilibrium, $\mu = \frac{1}{2}$.

It follows that $r \succ_2 l$ and then $y \succ_1 x$.