DYNAMIC MORAL HAZARD WITH SELF-ENFORCEABLE INCENTIVE PAYMENTS

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Preliminary and incomplete. Please do not cite.

ABSTRACT. This paper studies a dynamic moral hazard model in which the basic tool used to provide incentives to a risk-averse agent - higher payments for better outcomes - is limited by the principal's lack of commitment: good outcomes are not verifiable and hence the corresponding payments cannot be enforced by a court of law. The principal would like to drive down the value of the relationship to the agent indefinitely over time, as this alleviates both the incentive-provision and the enforceability problems. If the agent's utility has a lower bound, however, he faces a trade-off as there is less room to spread continuation values. In the optimal contract, the principal eventually chooses low effort or fires the agent. Incentive payments increase while the agent's continuation value decreases, as the former depend on the value of the relationship to the principal. When effort is continuous, the optimal contract has two phases: continuation values decrease on average before the lower bound is reached for the first time; and drift within an interval above it afterwards.

1. INTRODUCTION

Moral hazard calls for incentives to induce the agent to exert effort; however, good incentives frequently rely on measures that cannot be verified objectively by an outsider such as a court. Hence, these incentives cannot be enforced costlessly; if they are to be used, they must be self-enforced, meaning that one can only promise today what he shall be willing to give tomorrow.

Since self-enforceable contracts[1] can only be used in a repeated and open-ended relationship, a natural question is how incentives evolve over time. Under risk neutrality, it is known that optimal contracts can be made stationary: present and continuation payoffs may be exchanged at a one-to-one rate and therefore optimal incentives need not be made history-dependent[2]. However, repeated relationships often display some kind of non-stationarity: bonus to CEOs may increase over time and wages frequently depend on past performance. Such patterns arise naturally in environments where a risk-neutral principal must provide incentives for a risk-averse agent to exert unobservable effort.

[1]The type of contracts considered here has received many names: implicit, informal, self-enforceable, relational, sustainable, subgame perfect, non-verifiable. I will use "relational" or "self-enforceable" contracts and take these expressions as synonyms.

[2]See Levin (2003).

I consider a principal who cares about a project in which success cannot be verified by a court. Probability of success is higher when the agent makes some costly effort. The principal designs a bonus to the agent in case of success in order to provide incentive to work, but non-verifiability implies that this bonus must be self-enforceable.

If the agent cannot walk away from the relationship after signing the initial contract, the principal will drive his continuation value downwards indefinitely over time - in fact, it must diverge to minus infinity. This is traditional immiseration result and has two explanations in the present setup. The first is the moral hazard intuition: since the agent is risk-averse, his marginal utility is decreasing. At low levels of consumption, a small incentive payment is enough to induce high effort. The risk the agent bears is then lower; it follows that the cost of imperfect insurance decreases and the trade-off between efficiency and insurance is alleviated.

The second reason is related to the principal's lack of commitment. The non-verifiability problem implies that any incentive payment is limited by the future value of the relationship to him; if he promises a too-high bonus, the agent anticipates no payment will be made as the principal will prefer to give up the continuation of the relationship in order to have a current gain. This problem is lessened precisely when this continuation value is high: the set of credible promises then increases. To do so, the principal defers most of his own rewards to the future.

This declining trajectory amounts to an upfront payment scheme from the principal to the agent. Another issue is how the principal splits such payments between a formal wage (the enforceable base payment) and an informal bonus (the self-enforceable incentive payment). While the principal is restricted by self-enforceability, the incentive payment must increase over time. This follows exactly from the decrease in the agent's utility: as the relationship evolves, the principal gets more and more of the surplus generated. This way, the continuation of the relationship becomes more valuable to the principal as times goes by, implying that his incentive to renege on the bonus decreases. Hence, the bonus may increase for any level of effort the principal wants the agent to choose. Since total payment is decreasing, the enforceable base payment should decrease; it follows that formal and informal contracts are substitutes. As the agent's continuation value diverges to minus infinity, the self-enforceability constraint must eventually become moot. Afterwards, there is only a moral hazard problem; the principal will choose the unrestricted bonus, which should be decreasing exactly as the necessary gap in payments to induce effort is lower when the agent's consumption is low. Summing up, the bonus increases while self-enforceability is binding, and decreases when it becomes moot.

The ability of the principal to drive the agent's utility towards minus infinity is not appealing from an empirical perspective. The agent may have an outside option, or negative transfers from the princial may be impossible; in both cases, the agent's utility will have a lower bound.

I consider a case in which high effort is not optimal in the stage game; the principal needs the repetition of the relationships to spread rewards over time in order to achieve non-negative profits. In this case, memory is essential only if the principal wants to induce high effort. The principal still wants to decrease the agent's continuation value over a certain range; too close to the lower bound, however, he loses his ability to simultaneously spread and lower continuation values, so that he must turn to present consumption in order induce effort. However, present consumption is itself limited by his lack of commitment. Eventually it becomes too costly to provide incentives because he no longer has any tools, and he gives up high effort. Depending of the value of trade under low effort, he either fires the agent or induces low effort in every period afterwards: in the absence of incentive concerns, the contract disposes of memory and becomes stationary.

It is worth to emphasize that a higher bonus is associated with a *lower* utility for the agent. The only role of the bonus is to offer the agent the right incentives to work, while constrained by credibility and by the need to insure the agent at least partially. Notice that this result comes exactly from the interplay of risk-aversion, self-enforceability and hidden effort. If the agent were risk-neutral, the optimal contract could be made stationary from the beginning of the relationship; under observable effort, the bonus would be unnecessary; and with no commitment issues, the bonus would not increase systematically. The baseline is straightforward: bonuses increase incentives to work hard, but not necessarily personal well-being.

As mentioned, Levin (2003) showed that if the agent is risk-neutral, every non-stationary equilibrium has a payoff-equivalent stationary equilibrium. However, this indeterminacy does not survive if there is at least a small degree of risk-aversion. The optimal contract converges exactly to the stationary one if the agent's risk aversion converges to zero, which provides a selection mechanism for risk-neutral environments: the contracts considered in Levin (2003) are robust to small variations in risk aversion. The principal's value function becomes linear when the agent's utility is linear. The optimal contract then converges to the stationary one as there is no need to spread payoffs significantly.

A last point concerns the principal's ability to adjust effort. The results above can be seen as the principal's strategy to decrease the cost to provide incentives given the level of effort he wants the agent to exert. As this cost decreases, the principal will optimally increase this level if effort is continuous. This behavior, however, changes when the agent's continuation value approaches the lower bound: the principal must then decrease effort toward zero as it becomes harder to use continuation values. The contract is characterized by two stages. In the initial phase, continuation values decrease towards the lower bound; when they reach it for the first time (which is possible due precisely to the principal's ability to adjust effort), they drift above the lower bound indefinitely.

This paper belongs to the general class of dynamic moral hazard models. Lambert (1983) and Rogerson (1985) were the first to study the problem, and established the importance of memory to alleviate the insurance-efficiency trade-off. Spear and Srivastava (1987) solved the infinite-horizon problem, in which the memory property is even more relevant, and established the existence of a recursive structure[3]. Abreu, Pearce and Stacchetti (1990) proceeded to prove there is a recursive structure in a more general class of games. This paper also uses the recursive method, but the memory property that arises is less pervasive than in the seminal models.

[3]See also Thomas and Worrall (1990) for the adverse selection counterpart and Phelan (1995) for the problem with a lower bound.

Recent contributions have focused on financial contracting problems in a risk-neutral environment[4]. Clementi and Hopenhayn (2006) applied the moral hazard model to study firm dynamics. DeMarzo and Fishman (2007*a*) focused on investment dynamics, while DeMarzo and Fishman (2007*b*) considered long-term relationships. Biais et al. (2007) considered the implications of optimal security design for asset pricing. Biais et al. (2010) has some results closely related to the present paper. They also deal with an agent subject to limited liability, but since their model has another dimension to adjust (firm size), dynamics may be different: the firm may either grow indefinitely or shrink and disappear. The latter possibility occurs if there is a sequence with a high enough number of bad outcomes. Then the agent's continuation value gets too close to the lower bound and becomes harder to spread; incentive provision becomes too expensive and the principal will choose to downsize the firm, which may lead to liquidation. A similar result arises in the present model: if high effort is not optimal in the stage game, the principal eventually (but almost surely) chooses low effort - which is similar to "downsizing", but concerns only the task developed by the agent. Adjustment is in the control variable, not in the state.

The present paper adds to this literature by considering the impact of self-enforceability on the decisions made by the firm (i.e., the principal) and how the memory property is affected[5]. The risk-neutral environment considered in this papers, however, is not suitable for the study of self-enforceability in (potentially) history-dependent contracts due to the stationarity result in Levin (2003)[6]; as a corollary, adding a lower bound on the agent's utility is not suitable either. A second contribution is to allow for risk-aversion in self-enforceable contracts when provision of incentives is a concern[7]. Pearce and Stacchetti (1998) showed that enforceable and non-enforceable payments are substitutes: the latter increase when the principal's stake in the relationship increases, so that the agent's share decreases. A similar result arises in the present paper, but with an additional feature: due to incentive compatibility, the agent's stake decreases over time. Bernheim and Whinston (1998) also study the relationship between formal and informal contracts and established that the latter may be deliberately chosen ("strategic ambiguity") in order to allow for unverifiable information to be used in the contract. Such ambiguity is essential to provide incentives in the present model.

The paper proceeds as follows. The basic model is presented in section 2. Section 3 has the relevant benchmarks for later reference. Sections 4 solves the model with no lower bound, and section 5 describe the optimal contract under limited liability. Section 6 presents two extensions: the optimal contract under convergence to risk-neutrality and under continuous effort. Section 7 discusses some applications to compensation models and section 8 briefly concludes.

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[4]See Biais, Mariotti and Rochet (2011) for a recent review of this literaure.

[5]See MacLeod (2007) for a comprehensive review of the relational contracts literature.

[6]In an earlier exercise, Baker, Gibbons and Murphy (1994) assumed stationarity under risk-neutrality and showed, in a one-sided problem where only the agent can commit, that formal and informal contracts may be either complements or substitutes.

[7]Thomas and Worrall (1988) studied the trade-off between risk-sharing and self-enforcement in a complete information framework and showed that history-dependence arises in optimal contracts.

2. The model

Consider an infinitely repeated game between a principal and a risk-averse agent who share a common discount factor δ[8]. In each period they can engage in a spot contract in which the agent chooses an unobservable action $e \in \{e_l, e_h\}$ that determines the probability of success of a project that has value $\overline{Y}$ to the principal in case of success and $\underline{Y} < \overline{Y}$ in case of failure: $Prob\left(\overline{Y}\right) = p$ if $e = e_h$ (interpreted as high effort) and $\tilde{p} < p$ otherwise. The agent's Bernoulli utility function is separable in consumption and effort:

$$U(c, e) = u(c) - \psi(e)$$

in which $\psi(e_h) = K > 0$ and $\psi(e_l) = 0$. Assume $u(.)$ is twice differentiable, bounded above, strictly increasing and strictly concave.

To put aside issues related to the agent's savings, I will assume he has no access to credit (or, equivalently, the principal observes the agent's savings). Hence, income and consumption are used interchangeably. In order to induce the agent to exert effort, the principal must offer a higher payment in case of success: the contract determines $\overline{c}_t$ after success in period t and $\underline{c}_t$ after a failure such that $\overline{c}_t \geq \underline{c}_t$. $\underline{c}_t$ is a base salary and is enforceable by court. The difference $\overline{c}_t - \underline{c}_t$ is interpreted as a bonus in case of success in period t, but success is not verifiable and hence the bonus is not enforceable exogenously. Lastly, if the agent does not sign a contract with the principal, he gets his lifetime reservation utility u_{res} while the principal makes zero profit in a competitive market[9]. The principal solves:

$$\underset{\{c_t\}_{t=0}^{\infty}}{Max} \; E \sum_{t=0}^{\infty} \delta^t \left[y_t - c_t\right]$$

in which y_t and c_t are random variables such that $y_t \in \left\{\underline{Y}, \overline{Y}\right\}$ and $c_t \in \{\overline{c}_t, \; \underline{c}_t\}$. Instead of going over the constraints of this infinite-horizon problem, I will follow Spear and Srivastava (1987), who established that such games have a recursive formulation[10]. Let $w \geq u_{res}$ be the expected discounted future value of utility the principal must promise the agent in a given period and $V(w)$ be the principal's payoff; I will assume the initial value of w to be arbitrary, since it is not determined by the principal, but not as high as to render the contract unprofitable.

A recursive contract is a vector $\{\overline{c}, \underline{c}, \overline{w}, \underline{w}, e\}$, in which $\overline{w}$ and $\underline{w}$ are the continuation values the principal offers to the agent in case of success or failure, respectively. In order to implement high-effort (e_h), the principal's program may be written as:

$$V(w/e_h) = \underset{\overline{c}, \underline{c}, \overline{w}, \underline{w}}{Max} \; p\left[\overline{Y} - \overline{c} + \delta V(\overline{w})\right] + (1-p)\left[\underline{Y} - \underline{c} + \delta V(\underline{w})\right]$$

[8]As usual, this may be interpreted as a game with uncertainty about the end date.

[9]So if the agent's outside option is u per period, $u_{res} = \frac{1}{1-\beta} u$.

[10]Formally, this is a repeated game with imperfect public monitoring. A recursive structure was established in Abreu, Pearce and Stacchetti (1990). See also Mailath and Samuelson (2006) (chapter 7) for a proof.

subject to the following constraints:

$$p\left[u\left(\overline{c}\right)+\delta\overline{w}\right]+\left(1-p\right)\left[u\left(\underline{c}\right)+\delta\underline{w}\right]-K\geq\tilde{p}\left[u\left(\overline{c}\right)+\delta\overline{w}\right]+\left(1-\tilde{p}\right)\left[u\left(\underline{c}\right)+\delta\underline{w}\right]$$

$$\overline{Y}-\overline{c}+\delta V\left(\overline{w}\right)\geq\overline{Y}-\underline{c}$$

$$p\left[u\left(\overline{c}\right)+\delta\overline{w}\right]+\left(1-p\right)\left[u\left(\underline{c}\right)+\delta\underline{w}\right]-K=w$$

$$\overline{w},\underline{w}\geq u_{res}$$

The continuation value w will be used as the state variable for this problem. The first line is the usual incentive compatibility constraint (IC) for the agent: his expected utility when making effort must be higher than if he shirks. The second is the self-enforceability constraint (SE) that ensures the principal will pay the bonus $\overline{c}-\underline{c}$. Notice that if the principal reneges on the bonus, he gets the benefit from success and pays only $\underline{c}$, keeping $Y-\underline{c}$, and nothing else afterwards. I am implicitly assuming Nash-reversion after non-compliance by the principal, implying that bonus promises become immaterial. Then high effort can no longer be induced; the principal will either choose low effort or simply leave the relationship (depending on whether the fallback position $V\left(w/e_l\right)$ is greater than zero). Thus I am assuming away optimal punishment and renegotiation issues for the sake of simplicity, at a cost in terms of generality.

The third line is the promise-keeping (PK) constraint: it states that the principal must deliver the promised value w. Notice that he cannot give the agent more than w as this may be inconsistent with the promised values $\overline{w}$ and $\underline{w}$ in the previous period, implying that the incentive compatibility constraint would have failed to hold as the agent would anticipate this. It may also be interpreted as an ex-ante participation constraint: the (exogenous) initial value of w is the lowest value the agent accepts. The last constraint (LL) is necessary if the relationship is to keep going: it is a limited liability constraint stating that the agent cannot be forced to bear (expected) consumption levels below a threshold in any period. It may be interpreted as a participation constraint that must hold at the beginning of every period.

The problem above may be further simplified and rewritten as follows:

$$V\left(w/e_h\right)=\underset{\overline{c},\underline{c},\overline{w},\underline{w}}{Max}\;p\left[\overline{Y}-\overline{c}+\delta V\left(\overline{w}\right)\right]+\left(1-p\right)\left[\underline{Y}-\underline{c}+\delta V\left(\underline{w}\right)\right]$$

subject to

$$u\left(\overline{c}\right)+\delta\overline{w}-u\left(\underline{c}\right)-\delta\underline{w}-\frac{K}{p-\tilde{p}}\geq0$$

$$\underline{c}-\overline{c}+\delta V\left(\overline{w}\right)\geq0$$

$$p\left[u\left(\overline{c}\right)+\delta\overline{w}\right]+\left(1-p\right)\left[u\left(\underline{c}\right)+\delta\underline{w}\right]-K=w$$

$$\underline{w} - u_{res} \geq 0$$

History-dependence can be studied in terms of the state variable w. A stationary contract corresponds to a constant w: the state is then constant, and the principal will choose the same consumption and continuation value. A declining (increasing) w corresponds to upfront (backloaded) payments. It can also be interpreted as the evolution of bargaining power between the principal and the agent, since the total (expected) output of the relationship is fixed.

Notice that it is never optimal for the principal to to make $\overline{c} - \underline{c} < 0$, as established in the lemma below. Intuitively, he wants to use present payments in order to induce effort; the use of continuation values amounts to using current payments in a future period so as to spread the risk the agent bears.

Lemma 1. *The difference $\overline{c} - \underline{c}$ is non-negative.*

Before proceeding to the optimal contract, a preliminary step is to determine whether the incentive constraint (IC) is binding - were it not the case, the informational problem would be moot and have no interplay with self-enforceability. Lemma 2 below shows that this cannot be the case.

Lemma 2. *The incentive compatibility constraint is always binding.*

Hence moral hazard actually has a bite. Notice that this result holds irrespective of whether the other constraints bind or not (or whether they are even present in the problem): no other constraint drives the principal towards giving the agent a too-high utility in case of success. In fact, self-enforceability has exactly the opposite effect.

Notice that in the presence of self-enforceability and incentive compatibility, the constraint set will generally fail to be convex (unless the agent is risk neutral): both $u(\overline{c}) - u(\underline{c})$ and $\underline{c} - \overline{c}$ must be (at least weakly) concave functions of w. In order to guarantee this, the following assumption will be made:

Assumption 1. *The inverse of the agent's marginal utility is convex.*

Defining $h \equiv u^{-1}$, this simply means $h''' > 0$. The following result can then be established.

Proposition 1. *The value function $V(.)$ is strictly concave and differentiable.*

3. Benchmarks

3.1. First-Best Contract.

For future reference, consider initially the contract the principal would offer in the absence of any restrictions apart from an ex-ante participation constraint. There is only one present payment c and one continuation value w' conditional on high effort. The optimal contract solves the following program:

$$V(w/e_h) = \underset{c,w'}{Max} \quad \left[p\overline{Y} + (1-p)\underline{Y}\right] - c + \delta V(w)$$

subject to

$$u(c) + \delta w' - K \geq w$$

For any level of effort, the principal will always choose $\{c, w'\}$ such that (PK) holds with equality (notice that it does not need to be so *a priori*, as there is no incentive compatibility issue). Moreover, the agent does not bear any risk as there is no need to provide incentives. If the principal achieves positive profit with high effort for the initial value of w, he will never induce low effort. The optimal contract is stationary[11].

3.2. **The Repeated Moral Hazard Problem.** Assume now the principal cannot observe the agent's effort, but can commit to any promised payments. The agent's continuation value is not bounded below. The principal solves the program above with the additional incentive compatibility constraint:

$$V(w/e_h) = \underset{\overline{c},\underline{c},\overline{w},\underline{w}}{Max}\ p\left[\overline{Y} - \overline{c} + \delta V(\overline{w})\right] + (1-p)\left[\underline{Y} - \underline{c} + \delta V(\underline{w})\right]$$

subject to

$$u(\overline{c}) + \delta\overline{w} - u(\underline{c}) - \delta\underline{w} - \frac{K}{p - \tilde{p}} \geq 0$$

$$p\left[u(\overline{c}) + \delta\overline{w}\right] + (1-p)\left[u(\underline{c}) + \delta\underline{w}\right] - K = w$$

This is a simplified version of the problem studied by Spear and Srivastava (1987), and the resulting equilibrium is a particular case of the optimal contract they find. As established in lemma 1, (IC) must hold with equality. The problem may be rewritten as follows.

$$V(w/e_h) = \underset{\overline{c},\underline{c},\overline{w},\underline{w}}{Max}\ p\left[\overline{Y} - h\left(w + \overline{K} - \delta\overline{w}\right) + \delta V(\overline{w})\right] +$$

$$(1-p)\left[\underline{Y} - h\left(w + \underline{K} - \delta\underline{w}\right) + \delta V(\underline{w})\right]$$

The agent's continuation value will decrease on average: the principal drives the agent's continuation value towards a region where it is cheaper to provide incentives. When the agent faces low consumption, marginal utility (u') is high due to risk aversion ($u'' < 0$); hence a small difference between $\overline{c}$ and $\underline{c}$ is enough to induce effort, without transferring much risk to the agent. The principal can induce effort with little distortion in terms of imperfect risk-sharing. Moreover, there is no lower bound: if it were so, either continuation values would converge (so that the principal would be inducing effort solely through present payments) or they would eventually converge to a non-degenerate distribution above the lower bound. Both these possibilities are worse for the principal than letting

[11]Actually, there are payoff-equivalent non-stationary equilibria. For example, the principal could make an upfront payment and then drive the agent's continuation value to u_{res}.

w float downwards towards minus infinity. Again, the principal will always choose high effort if it is profitable for the initial value of w - it will only become cheaper to provide incentives as times goes by and the agent's continuation value decreases. Define w' as the value of the state variable in the beginning of the next period. One has the following corollary of Spear and Srivastava (1987) and Thomas and Worrall (1990).

Proposition 2. *(Immiseration Result) For all w, $w > E(w')$: on average, the agent's continuation value decreases over time. With probability one, it diverges towards minus infinity*[12].

As for the bonus $\bar{c} - \underline{c}$, the argument above implies that it will eventually converge to zero: a lower difference is needed when the agent's marginal utility is low. The principal will be providing incentives through continuation values, which are linear in the agent's utility. Due to the fact that incentive compatibility is always binding, the higher $\overline{w} - \underline{w}$, the lower $u(\bar{c}) - u(\underline{c})$[13].

3.3. **Self-Enforceability.** A less clear setup happens when the only restriction on top of the ex-ante participation constraint is self-enforceability, which places a limit on the size of the bonus the principal can credibily promise. The problem looks as follows:

$$V(w/e_h) = \underset{\bar{c},\underline{c},\overline{w},\underline{w}}{Max}\ p\left[\overline{Y} - \bar{c} + \delta V(\overline{w})\right] + (1-p)\left[\underline{Y} - \underline{c} + \delta V(\underline{w})\right]$$

subject to

$$\underline{c} - \bar{c} + \delta V(\overline{w}) \geq 0$$

$$p\left[u(\bar{c}) + \delta\overline{w}\right] + (1-p)\left[u(\underline{c}) + \delta\underline{w}\right] - K \geq w$$

However, the solution to this problem is trivial. Effort is observable and hence there is no need to provide incentive for the agent to privately choose high effort; it follows that the principal will never put a gap in payments. The first-best contract is available and the self-enforceability constraint is necessarily moot; it will only matter when there is an interplay with incentive compatibility.

There is however an alternative way to set up this problem. Assume, as in Pearce and Stacchetti (1998), that both the principal and the agent observe the choice of effort, but not a third party - a court that could enforce the contract. Then the principal promises a bonus conditional on high effort (not on high output, as in moral hazard problems), but this bonus must be self-enforced as the court of law cannot observe whether the agent chose high effort - only the two parties within the relationship. The solution to this problem has some important similarities with the present paper, as will be seen below.

[12]In the present model, this is a particular case of proposition 5, so only the proof for the latter is provided.

[13]Too see this strictly, assume for the moment $h''' = 0$, and notice that $w_l' - w_h' = \frac{h_h' h_h'' - h_l' v_h'' + h_h' v_h''}{\delta h_l' \left[\delta h_h'' - v_h''\right]}$. Only the last term on the left-hand side is negative, but it must converge to zero as $w \to -\infty$.

4. Immiseration Result Revisited

In order to start studying the relationship between incentive compatibility and self-enforceability, consider an initial setup without a limited liability constraint - i.e., the principal is allowed to drive the agent's continuation value as low as he wants. The principal must solve the following problem in order to induce high effort:

$$V(w/e_h) = \underset{\overline{c},\underline{c},\overline{w},\underline{w}}{Max}\ p\left[\overline{Y} - \overline{c} + \delta V\left(\overline{w}\right)\right] + (1-p)\left[\underline{Y} - \underline{c} + \delta V\left(\underline{w}\right)\right]$$

subject to

$$u\left(\overline{c}\right) + \delta\overline{w} - u\left(\underline{c}\right) - \delta\underline{w} - \frac{K}{p-\tilde{p}} \geq 0$$

$$\underline{c} - \overline{c} + \delta V\left(\overline{w}\right) \geq 0$$

$$p\left[u\left(\overline{c}\right) + \delta\overline{w}\right] + (1-p)\left[u\left(\underline{c}\right) + \delta\underline{w}\right] - K \geq w$$

This is simply a repeated moral hazard problem in which the current-consumption bonus the principal can credibly promise is limited. However, this does not modify the long-term behavior of continuation values: proposition 2 is not affected.

Proposition 3. *The immiseration result holds unchanged in the infinitely-repeated moral hazard model with a self-enforceability constraint on the principal*[14].

In the absence of a lower bound on continuation values, incentive compatibility and self-enforceability effects go in the same direction. As in the pure moral hazard model, the principal wants to decrease the agent's continuation value in order to make the provision of incentives cheaper. This effected is strengthened by self-enforceability: the non-commited party (the principal) accumulates future payoffs so that the continuation of the relationship is more valuable to him, and hence his incentive to renege decreases - thus the (SE) constraint is relaxed.

The usual long-term result in the pure repeated moral hazard literature arises again: the agent's continuation value goes to minus infinity over time. Intuitively, the principal has no reason to cap below the agent's continuation value, since the agent is bound to the contract. Formally, the sequence $\{V'(w)\}$ is a non-positive submartingale and must converge eventually; it cannot converge to any finite number as the principal would be ceasing to use continuation values to provide incentives exactly when they are cheaper, which is suboptimal. It follows that it must converge to minus infinity. This has one relevant implication: eventually, the self-enforceability constraint must become moot. The principal is then back to a pure moral hazard problem.

The behavior of the bonus then depends on the "stage" of the contract - i.e., whether the self-enforceability constraint binds or not. If it does not bind, then the bonus will decrease on average, as described in the previous section. If it binds, the bonus is determined by the following inequality:

[14]Again, this is a particular case of proposition 5.

$$\overline{c} - \underline{c} = \delta V(\overline{w})$$

The bonus is determined by two factors: the behavior of $\overline{w}$ and the slope of the value function V. Consider first the former: a lower value of w means that the principal needs to give less average consumption to the agent - whether in present or future payments. In particular, he will decrease the agent's continuation value in case of success, as established below.

Lemma 3. *$\overline{w}$ is increasing in w.*

Consider now the slope of the value function V. In this problem, it simply describes the efficient payoff frontier: the agent's payoff increases if and only if the principal's rent decreases.

Lemma 4. *The value function V is strictly decreasing.*

The next result follows immediately.

Proposition 4. *If the self-enforceability constraint is binding, the bonus $\overline{c} - \underline{c}$ is negatively correlated with the agent's continuation value w.*

A binding (SE) implies $\overline{c} - \underline{c} = \delta V(\overline{w})$; the left-hand side is simply the bonus. Then $V' < 0$ and decreasing continuation values to the agent imply that it increases (on average) over time. Decreasing continuation values mean the firm will have more at stake in the next period and the self-enforceability constraint will be less tight. Again, the fact that the agent's utility goes down, so that the firm gets more and more of the net benefit generated in the relationship, implies that the principal will be able to make better use of the bonus for any given level of effort - hence the bonus increases over time.

Self-enforceability puts a significant restriction on the principal's ability to design the optimal bonus; hence it imposes a lot of structure on it. Note that formal and informal contracts are substitutes: while $\overline{c} - \underline{c}$ goes up, $\underline{c}$ decreases[15]. It is worth highlighting that this result has a similar counterpart in Pearce and Stacchetti (1998), and the driving force is the same: "bonuses smooth the consumption path of the risk-averse agent by moving in the opposite direction from salaries, total consumption, and expected discounted utility for the rest of the game". As discussed above, they do not deal with incentives to effort, which is observed by both parties: the driver of a bonus negatively related to the agent's continuation value is not the unobservability of effort, but the principal's lack of commitment. The novelty is that now it has a time pattern.

Propositions 3 and 4 completely characterize the behavior of the bonus: on average, it decreases while the self-enforceability constraint is binding, and starts decreasing when it becomes slack. Once (SE) becomes moot, the principal will choose the optimal unrestricted bonus in every period, and the problem becomes a repeated moral hazard one.

[15]An increasing bonus means $d(\overline{c} - \underline{c}) > 0$, and hence $d\overline{c} > d\underline{c}$. The marginal rate of substitution between $\overline{w}$ and $\overline{c}$ implies $\frac{d\overline{c}}{d\overline{w}} > 0$, while proposition 3 and lemma 3 has $d\overline{w} < 0$. Then $0 > d\overline{c} > d\underline{c}$. This just reflects the agent's decreasing continuation values.

Intuition is straighforward: the only limit to the bonus is the continuation value of the relationship to the principal.

Notice lastly that if high effort is optimal for the initial continuation value of w (as assumed), it must be optimal for every continuation value afterwards as it will only become cheaper for the principal to provide incentives[16]. The interaction between (IC) and (SE) adds structure to the bonus - which will be decreasing up to some point, and then decreasing (both on average). If (IC) were absent, the bonus and continuation values would move in opposite directions, but with no trend. If (SE) were absent, the bonus would decrease indefinitely. The bottomline is that a higher bonus is not associated to a higher continuation value to the agent: its role is to induce effort, not increase any measure of utility, and it rises as a response to a less tight self-enforceability constraint - which happens when the principal's value is higher. However, when it comes to the behavior of continuation values, self-enforceability and incentive compatibility go in the same direction. The next section turns to the case where (IC) and (SE) do not have the same impact on continuation values.

5. Optimal Contract under Limited Liability

Consider now the full-fledged problem in which the principal is subject to the limited liability constraint. From now on, assume that self-enforceability is binding in the relevant region: the unrestricted bonus is not credible for any w in the interval $[u_{res}, w_0]$, in which w_0 is the initial (exogenous) value of w - i.e., just asssume that the optimal static bonus is not credible. To avoid corner solutions, assume also that $u\left(\right)$ satisfies the Inada conditions[17]. Lastly, this program yields a strictly positive payoff to the principal for the initial value of w.

Consider initially the contract that induces low effort. Since the principal does not need to provide incentive to effort, it is not necessary to make the agent's current or future payments conditional on output. The lemma below then follows.

Lemma 5. *The contract that induces low effort is stationary.*

As a corollary, low effort is chosen in every period afterwards.

Turn now to the optimal contract that induces high effort. Since the problem is strictly concave, the Kuhn-Tucker conditions are sufficient to characterize the solution. Let γ, λ, μ and η be the multipliers of the four constraints, respectively. The first-order conditions are:

FOC($\overline{c}$): $-p + \gamma u'\left(\overline{c}\right) - \lambda + \mu p u'\left(\overline{c}\right) = 0$

[16]There is a caveat to this argument: a long enough sequence of good outcomes at the beginning of the relationship could make the agent's continuation value too high for the principal to profit from the relationship. It can be shown, however, that the principal will never drive the agent into such a region.

[17]If consumption is bound to be non-negative, this boils down to three conditions:

1- $u\left(0\right) = 0$

2- $\lim\limits_{c \to 0} u'\left(c\right) = \infty$

3- $\lim\limits_{c \to \infty} u'\left(c\right) = 0$

FOC($\underline{c}$): $-(1-p) - \gamma u'(\underline{c}) + \lambda + \mu(1-p)u'(\underline{c}) = 0$
FOC($\overline{w}$): $\delta p V'(\overline{w}) + \gamma\delta + \lambda\delta V'(\overline{w}) + \mu p = 0$
FOC($\underline{w}$): $\delta(1-p)V'(\underline{w}) - \gamma\delta + \mu(1-p)\delta + \eta\delta = 0$
Envelope theorem: $V'(w) = -\mu$

Notice initially that FOC($\overline{c}$) and FOC($\underline{c}$) imply the following expression for μ[18]:

$$\mu = \frac{1 + \gamma\left[u'(\underline{c}) - u'(\overline{c})\right]}{pu'(\overline{c}) + (1-p)u'(\underline{c})}$$

which is strictly positive due to $u' > 0$, $u'' < 0$ and $\overline{c} \geq \underline{c}$. Hence the value function conditional on high effort (i.e., $V(w/e_h)$) is strictly decreasing. The slope of the value function decreases (in absolute value) due to the fact that he loses a tool to induce effort as he approaches the lower bound u_{res}. To see this, notice first that the difference $u'(\underline{c}) - u'(\overline{c})$ decreases due to assumption 1. Moreover, if $\eta = 0$, then $\gamma = V'(\underline{w}) - V'(w)$; this difference must converge to zero close to the lower bound u_{res} if $\underline{w} \leq w$[19]. Lastly, the denominator increases as the marginal utility is increasing and average consumption is increasing in w. However, V' does not become increasing as long as the principal does not adjust effort.

In order to see the distortion introduced by self-enforceability, it is enough to check the relationship between marginal rates of substitution. First-order conditions for $\overline{c}$ and $\overline{w}$ imply the following equality:

$$u'(\overline{c}) = \frac{-1}{V'(\overline{w})} \tag{5.1}$$

This is simply the equality between the agent's marginal rate of substitution (evaluated at $\overline{c}$) and the principal's marginal rate of transformation (evaluated at $\overline{w}$). Notice that it does not depend on λ - in fact, it is exactly the same relationship that holds in repeated moral hazard models. It implies that $\overline{c}$ and $\overline{w}$ have a positive relationship. A similar computation for $\underline{c}$ and $\underline{w}$ yields the following result:

$$\frac{\lambda}{1-p} = 1 + V'(\underline{w})\,u'(\underline{c}) \tag{5.2}$$

Thus self-enforceability ($\lambda > 0$) puts a constraint on the relationship between $\underline{c}$ and $\underline{w}$: the restriction on the bonus is implemented through a higher $\underline{c}$.

The next step is to describe the dynamics of the optimal contract, i.e., to estabilish the behavior of continuation values in equilibrium with respect to the initial promised value w. The following proposition shows that as long as the limited liability constraint is slack, the evolution of the relationship follows that same "immiseration" pattern found in the previous section.

[18]Notice that this expression does not depend neither on λ nor on η: it holds whether (SE) and (LL) are present or not.
[19]It may be shown that $w < \underline{w}$ is necessarily suboptimal, as there is another contract with $\underline{w} \leq w$ such that the constraints are respected and the principal's profit is higher.

Proposition 5. *Assume (LL) is slack (i.e., $\eta = 0$) and the principal implements high effort (e_h). Then the agent's continuation value will decrease on average: $w > E(w')$.*

Without restrictions on the technology, the principal will generally make $\overline{w} > w > \underline{w}$ such that the continuation value w will decrease only on average.

In principle, there are many combinations of present consumption and continuation values that could work; the behavior of the bonus $\overline{c} - \underline{c}$ has not been determined. The presence of self-enforceability, however, allows one to determine the time structure of the optimal bonus: it is an immediate consequence of propositions 4 and 5.

Corollary 1. *Assume (SE) is binding (i.e., $\lambda > 0$), (LL) is slack (i.e., $\eta = 0$) and the principal implements high effort (e_h). Then the bonus $\overline{c} - \underline{c}$ increases on average over time.*

The bonus will follow a similar pattern as in the previous section as long as (LL) is slack: it will increase on average over time. To complete the characterization of the contract, assume now that it is not optimal to induce high effort after at least some histories (i.e., for some continuation values). This implies that it is not optimal to induce high effort in the stage game, when the cost of inducing effort is highest as the principal cannot use continuation values to spread risk. This is equivalent to assuming that it is not optimal to induce high effort in the stationary dynamic game.

$$V(e_h/u_{res}) \leq 0 < V(e_l/u_{res}) \tag{5.3}$$

Then eventually it is optimal for the principal to give up high effort[20]. The next proposition establishes this point.

Proposition 6. *The principal induces low effort when continuation values are close enough to the lower bound u_{res}.*

This implies that there is a threshold w^* such that:

$$V(w^*/e_h) = V(w^*/e_l)$$

After some time, it becomes too costly to provide incentives for the agent to exert effort. The principal would like the agent to exert high effort in every period, following any history. To do so, however, he must incur a second-best cost due to incentive compatibility. He can decrease this cost by spreading punishments and rewards over time - i.e., but putting a gap both in present consumption and in continuation values -, and decreases the agent's continuation value over time to lower it further. Close to the lower bound, however, he loses his ability to spread and lower continuation values simultaneously, so that he must turn to present consumption in order induce effort. However, present consumption is itself limited by his lack of commitment. Lastly, the principal cannot return

[20]An analogous point can be made if $V(e_l/u_{res}) < 0$: then the principal will leave the relationship instead of inducing low effort.

to the initial promised value to the agent as the promise-keeping constraint must hold with equality - otherwise intertemporal incentive compatibility would be lost. All considered, the principal eventually gives up high effort: when it becomes too costly to provide incentives because he no longer has any tools. The more failures the agent gets, the sooner this will happen; but it will eventually happen for every agent with probability one.

Notice lastly that since the value function is continuous (a corollary of differentiability), the value function under low effort $(V(w/e_l))$ must be steeper than the value function under high effort $(V(w/e_h))$ when continuation values are close enough to the lower bound. In fact, the difference (in absolute value) is equal to $-\delta(w-\underline{w})+\frac{p}{\Delta p}$, which must become negative close enough to the lower bound. This implies that the principal would like to randomize between low and high effort within an interval of continuation values around the cutoff w^*.

It is interesting to compare this result to Biais et al. (2010). They find that when continuation values are too low and it becomes too expensive to provide incentives to effort due to the lower bound, the principal will choose to downsize the firm. In the present setup, instead of downsizing, the principal gives up effort - which may also be interpreted as destroying value as an optimal response to a very high second-best cost of providing incentives to high effort. While they adjust a second state variable, in the present setup the main control variable is used by the principal.

6. Extensions

6.1. **Risk Neutrality.** The results presented in the previous sections hinge decisively on the agent's risk aversion. As mentioned in the introduction, under risk neutrality it was shown that if there is an equilibrium, there is a stationary equilibrium equivalent to it (Levin (2003)). One natural question is then what happens when risk aversion approaches zero: does the optimal contract approach the stationary benchmark or some other (payoff-equivalent) equilibrium?

Proposition 7. *The optimal contract converges to the stationary one as the agent becomes risk neutral.*

This result may be split into two parts. First, if the agent is risk neutral, the principal's value funcion becomes linear: the principal is risk-neutral and only benefits from spreading payoffs as this provides incentives for the agent to exert effort at a cost in terms of imperfect insurance. If there is no such cost, the way payoffs are split is irrelevant. Second, the optimal contract is stationary when the value function is linear: the principal makes an upfront payment, drives the agent down to his reservation continuation value u_{res}, and no longer changes the contract. Average consumption afterwards is such that:

$$u(c)=\frac{u_{res}}{1-\delta}$$

Again, the only value of history-dependence is to drive the agent's continuation value downwards; once this is no longer possible, there is no need to make present consumption depend on past performance. The optimal contract is simply determined by the four constraints.

6.2. **Continuous Effort.** The previous sections assumed the principal is very limited in adjusting effort. They can be interpreted as if he must guarantee a given level of effort, with shutdown as the only alternative. Assume now that effort $e \in [0,1]$ is continuous and that the cost function $K(e)$ is twice differentiable and such that $K'(e) > 0$, $K''(e) > 0$, $K(0) = K'(0) = 0$ and $K'(1) = \infty$. The probability of success is assumed to be $p(e) = e$. The rest of the structure of the game is the same as in section 6; in particular, it is not optimal for the principal to induce any positive level of effort in the stage game.

Using the first-order approach, the program may be stated as follows.

$$V(w) = \underset{\overline{c},\underline{c},\overline{w},\underline{w},e}{Max} \; e\left[\overline{Y} - \overline{c} + \delta V(\overline{w})\right] + (1-e)\left[\underline{Y} - \underline{c} + \delta V(\underline{w})\right]$$

subject to

$$u(\overline{c}) + \delta\overline{w} - u(\underline{c}) - \delta\underline{w} - K'(e) = 0$$

$$\underline{c} - \overline{c} + \delta V(\overline{w}) \geq 0$$

$$e\left[u(\overline{c}) + \delta\overline{w}\right] + (1-e)\left[u(\underline{c}) + \delta\underline{w}\right] - K = w$$

$$\underline{w} - u_{res} \geq 0$$

One can use (IC) and (PK) to rewrite this problem without consumption levels:

$$V(w) = \underset{\overline{w},\underline{w},e}{Max} \; e\left[\overline{Y} - h\left(K(e) + (1-e)K'(e) + w - \beta w_h\right) + \delta V(\overline{w})\right] +$$

$$+ (1-e)\left[\underline{Y} - h\left(K(e) - eK'(e) + w - \beta w_l\right) + \delta V(\underline{w})\right]$$

subject to

$$h\left(K(e) - eK'(e) + w - \beta w_l\right) - h\left(K(e) + (1-e)K'(e) + w - \beta w_h\right) + \delta V(\overline{w}) \geq 0$$

$$\underline{w} - u_{res} \geq 0$$

The agent's continuation value w still decreases as long as the limited liability constraint does not bind[21]. However, the limited liability constraint may now bind as the principal has an additional degree of freedom - effort - to adjust. Hence (LL) will eventually bind, as it is optimal for the principal to decrease the agent's share.

When the principal can adjust effort smoothly, the optimal contract has two phases. Before (LL) binds for the first time, continuation values decrease on average over time: $w > E(w')$. This is the same behavior found in the previous sections. After (LL) binds for the first time, however, w does not decrease on average anymore. To see this, notice that when $\eta > 0$, the proof of proposition 5 breaks down and one may have $w < E(w')$; continuation values then drift above the lower bound u_{res}. If the agent has a sequence of

[21]The proof of proposition 5 goes unchanged.

positive outcomes, w may increase above the region where (LL) binds; then it will start decreasing again.

Effort follows a non-monotonic pattern. To see this, notice that due to the agent's first-order condition and $K'' > 0$, the equilibrium level of effort is an increasing function of $u(c_h) - u(c_l) + \beta(w_h - w_l)$. In the initial phase, the principal can decrease continuation values in order to make it cheaper to provide incentive to effort. Section 5 can be seen as the principal choosing the cheapest contract to induce a given level of effort. A lower continuation value to the agent will increase the continuation value of the relationship to the principal for any such level, lowering the principal's incentive to renege on the bonus; he can then credibly promise a higher bonus, and the optimal level of effort increases. Formally, $u(c_h) - u(c_l)$ goes up while $(w_h - w_l)$ is essentially unchanged. Putting it differently, the principal cannot choose a level of effort as high as he would like as of the first period as this would entail a too-high (and therefore non-credible) bonus; he is forced to lower effort in the initial stages, and raises it over time as he gains room to make promises.

This pattern changes when the continuation value approaches the lower bound. Notice initially that if $w_h = w_l$, then effort must be equal to zero as it is not optimal for the principal to induce positive effort in the stage game. It follows that the principal must eventually start decreasing the optimal level of effort as continuation values decrease. In the second phase of the contract, as the continuation value w fluctuates, so does effort.

The next proposition summarizes these results.

Proposition 8. *In the model with continuous effort, the agent's continuation value decreases on average ($w > E(w')$) before the limited liability binds for the first time, and then drifts above the lower bound u_{res}. Moreover, there is a cutoff w^c such that effort is decreasing in w if and only if $w < w^c$.*

7. Compensation Contracts

The framework above has straighforward applications in different economic contexts. In empirical terms, the main result concerns the time-pattern of self-enforceable payments: history-dependence is "strong" in the initial stages and softens down as the relationship evolves.

Prendergast (1999) discusses at length, from an empirical point of view, how long-term employer-employee relationships differ from static ones. In other words, real-world contracts are more often than not non-stationary. The specific time-pattern, however, depends on the problem at hand. The present paper adds to this literature by putting some structure on this pattern.

When the agent's outside option is valuable (u_{res} is high compared to $u\left(\frac{Y}{1-\delta}\right)$), the early phase of the optimal contract tends to be short as the optimal contract cannot be far from the lower bound. Hence the model predicts history-dependence to be less relevant in there is competition among employers for a worker. A similar reasoning applies to the agent's bargaining power at the beginning of the relationship (reflected in the initial value of w): when it is large, contracts should be highly non-stationary. Hence the current performance of workers with very specific abilities should have a significant impact of future compensation. Workers with no bargaing power should be offered contracts closer to the stationary benchmark; as a corollary, bonuses for good outcome should not increase significantly over time.

The effort pattern predicted by the model also depends on the initial value of w. If the agent's bargaining power is too high at the beginning of the relationship, the principal will not be able to elict much effort in the first periods: effort will increase over time until it starts decreases. If the initial bargaing power is not too high, the principal will begin the relationship with maximum effort and decrease it over time until it starts to float with no trend; workers with no specific abilities are then predicted to enter the relationship at the highest level of effort they will be induced to exert.

The second stage of the contract depends on the room the principal has to adjust the agent's effort. If he is required to keep a specific level of effort, the contract converges to the stationary benchmark. If the required level of effort does not guarantee positive profits indefinitely, the agent is eventually pulled away from his duty. This is the case when job stability is high and the employer has little room to choose the employee. Civil service is an example: workers are often assigned to a given superior and cannot be fired.

Lastly, it is important to highlight that all the results rely on the assumption that the agent's outside option is fixed: u_{res} does not depend on w. This may be interpreted as an employer who deals with a very mechanical competition for the agent. It is an open question how the results above would be affected by strategic competition, in which other principals would take into account the current continuation value promised to the agent.

8. Final remarks

This paper describes the solution to a moral hazard problem in which the very tool used to balance incentives and insurance is limited by the lack of exogenous enforceability. There is room for extending the analisys in different directions. First, I assumed that the agent plays a grim-trigger strategy: after the principal reneges on the deal, there is immediate Nash-reversion. This is the best scenario for the principal to use self-enforceable payments. However, in equilibria with renegotiation after breach of contract, inducing effort is still more expensive; this should affected mostly the early stages of the contract. Second, I assumed that the principal faces no strategic competition for the agent: he can drive the continuation value w without affecting u_{res}. A relevant extension is to allow for competing principals.

On an empirical ground, the model can be used to draw patterns about optimal compensation in labor markets - in fact, the recent dynamic moral hazard literature has shown a strong focus on corporate finance and CEO compensation. In particular, different parameters of risk-aversion can generate different time patterns for bonuses, wages and effort; a numerical exercise would help visualize such patterns.

9. Appendix

Proof. Lemma 1

If the principal wants to induce low effort, then there is no need to transfer risk to the agent; hence $\overline{c} - \underline{c} = 0$. If he wants to implement high effort and (SE) is binding, then $\overline{c} - \underline{c} = \delta V(\overline{w}) \geq 0$ as the principal's discounted profit cannot be negative. If (SE) is not binding after some history, then the problem becomes a pure repeated moral hazard one for some value $\hat{w}$ of the state variable. It is then a particular case of the model presented in **?** and it follows from their characterization of the optimal contract (section 8.2, p. 335) that $\overline{c} - \underline{c} \geq 0$ at $\hat{w}$. In the present setup, the repeated moral hazard problem is also strictly concave and differentiable; one may define γ and μ as the respective multipliers of (IC) and (PK) and rearrange the first-order conditions to get:

$$\gamma\left[u\left(\overline{c}\right) - u\left(\underline{c}\right)\right] = -1 - \mu E\left[u'\right] < 0$$

in which $E\left[u'\right] \equiv pu'\left(\overline{c}\right) + (1-p)\,u'\left(\underline{c}\right)$. The strict concavity of u then implies $\overline{c} - \underline{c} > 0$. □

Proof. Lemma 2

Assume (IC) is not binding:

$$u\left(\overline{c}\right) + \delta\overline{w} - u\left(\underline{c}\right) - \delta\underline{w} - \frac{K}{\Delta p} > 0$$

Then it is possible to construct a new contract that increases the principal's profit and the other constraints are not violated: decrease high consumption by $d\overline{c} < 0$ and increase low consumption by $d\underline{c} > 0$. Change in profits will be:

$$-pd\overline{c} - (1-p)\,d\underline{c}$$

Since (PK) should still be binding, these changes must be such that $pu'\left(\overline{c}\right)d\overline{c} + (1-p)\,u'\left(\underline{c}\right)d\underline{c} = 0$, which implies:

$$d\overline{c} = -\left(\frac{1-p}{p}\right)\left(\frac{u'\left(\underline{c}\right)}{u'\left(\overline{c}\right)}\right)dc_l$$

It follows that the change in profit will be $(1-p)\,d\underline{c}\left[\frac{u'(\underline{c})}{u'(\overline{c})} - 1\right] > 0$. The initial contract with a slack (IC) could not be optimal. □

Proof. Proposition 1

Define $u\left(\overline{c}\right) \equiv \overline{u}$ and $u\left(\underline{c}\right) \equiv \underline{u}$. Rewrite (IC) and (PK) to get: $\overline{u} = w + \overline{K} - \delta\overline{w}$ and $\underline{u} = w + \underline{K} - \delta\underline{w}$, in which $\overline{K} = \frac{1-p}{\Delta p} + 1$ and $\underline{K} = 1 - \frac{p}{\Delta p}$ (notice that $\overline{K} > \underline{K}$) . Let $h \equiv u^{-1}$ and define the operator T as follows:

$$TV(w) = \underset{\overline{w},\underline{w}}{Max}\ \ p\left[\overline{Y} - h\left(w + \overline{K} - \delta\overline{w}\right) + \delta V\left(\overline{w}\right)\right] +$$

$$(1-p)\left[\underline{Y}-h\left(w+\underline{K}-\delta \underline{w}\right)+\delta V\left(\underline{w}\right)\right]$$

subject to

$$h\left(w+\underline{K}-\delta \underline{w}\right)-h\left(w+\overline{K}-\delta \overline{w}\right)+\delta V\left(\overline{w}\right)\geq 0$$

$$\underline{w}-u_{res}\geq 0$$

The difference $h\left(w+\underline{K}-\delta \underline{w}\right)-h\left(w+\overline{K}-\delta \overline{w}\right)$ is concave in w since $h'''>0$ and $\overline{u}\geq \underline{u}$. Assume that V is concave; then the LHS of (SE) is concave. Since (LL) is linear, it follows that the constraint set is convex. The objective function is also strictly concave due to $-h''<0$. It follows from theorem 4.6 in Stokey, Lucas and Prescott (1989) that the operator T is strictly concave, and hence the value funcion V is strictly concave. Differentiability follows from theorem 4.11.

□

Proof. Lemma 3

Simply apply the implicit function theorem to the first-order conditions of the problem to get:

$$\frac{d\overline{w}}{dw}=-\frac{h'\left(\overline{u}\right)}{V''\left(\overline{w}\right)-\delta h''\left(\overline{u}\right)}>0$$

□

Proof. Lemma 4

Notice initially that proposition 1 applies to the program without (LL). Defining γ as the multiplier of the (IC) constraint and using the Kuhn-Tucker conditions and the envelope theorem, one gets $V'\left(w\right)=-\mu=-\frac{1+\gamma\left[u'\left(\underline{c}\right)-u'\left(\overline{c}\right)\right]}{pu'\left(\overline{c}\right)+(1-p)u'\left(\underline{c}\right)}<0$.

□

Proof. Lemma 5

□

In order to induce low effort, the principal solves the following problem:

$$V\left(w\right)=\underset{w'}{Max}-h\left[w+\underline{K}-\delta w'\right]+\delta V\left(w'\right)$$

subject to $w'-u_{res}\geq 0$, in which $h=u^{-1}$. Let η be the multiplier of this constraint, and notice that this program is trivially concave. The first-order condition and the envelope theorem state, respectively:

$h'\left[w'+\underline{K}-\delta w\right]+V'\left(w'\right)+\eta=0$
$V'\left(w\right)=-h'\left[w'+\underline{K}-\delta w\right]$

Assume $\eta>0$ and $V"<0$. Then $V'\left(u_{res}\right)<V'\left(w\right)$ and hence $u_{res}>w$, which is absurd. Hence the concavity of V implies $\eta=0$. In this case, one has $w=w'$ and the problem is the same is the next period; hence if e_l is optimal today, it will be optimal tomorrow.

Proof. Proposition 5

Consider again the problem as set in proposition 1 and assume (LL) is slack. First-order conditions with respect to $\overline{w}$ and $\underline{w}$ are:

$(p+\lambda)\left[h'\left(w+\overline{K}-\delta\overline{w}\right)+V'\left(\overline{w}\right)\right]=0$
$(1-p)\left[h'\left(w+\underline{K}-\delta\underline{w}\right)+V'\left(\underline{w}\right)\right]-\lambda h'\left(w+\underline{K}-\delta\underline{w}\right)=0$

To simplify notation, define $\overline{h}'\equiv h'\left(w+\overline{K}-\delta\overline{w}\right)$ and $\underline{h}'\equiv h'\left(w+\underline{K}-\delta\underline{w}\right)$. Adding up the two equalities above, one gets the following expression for $E\left(V'\right)\equiv pV'\left(\overline{w}\right)+(1-p)\,V'\left(\underline{w}\right)$:

$$E\left(V'\right)=-\lambda V'\left(\overline{w}\right)-p\overline{h}'-\lambda\overline{h}'-\underline{h}'+p\underline{h}'+\lambda\underline{h}'$$

From the envelope theorem, one has the following:

$$V'\left(w\right)=-p\overline{h}'-\lambda\overline{h}'-\underline{h}'+p\underline{h}'+\lambda\underline{h}'$$

Hence $E\left(V'\right)>V'\left(w\right)$, establishing that V' is a non-positive submartingale. The concavity of the value function then implies $V'\left(E\left(w'\right)\right)>E\left(V'\left(w'\right)\right)>V'\left(w\right)$ and lastly $w>E\left(w'\right)$.

□

Proof. Proposition 6

In order to simplify notation, define $V_h\left(w\right)\equiv V\left(w/e_h\right)$, $V_l\left(w\right)\equiv V\left(w/e_l\right)$, $E_h\left(Y\right)\equiv p\overline{Y}+(1-p)\,\underline{Y}$ and $E_l\left(Y\right)\equiv\tilde{p}\overline{Y}+(1-\tilde{p})\,\underline{Y}$. From lemma 5, $V\left(w/e_l\right)$ can be defined as:

$$V_l\left(w^*\right)=-h\left[w^*\left(1-\delta\right)+\underline{K}\right]+\delta V_l\left(w^*\right)$$

The last term on the right-hand side is V_l because the contract that induces low effort is stationary. The value w^* is determined by $V'\left(w^*\right)=-h'\left(w^*\left(1-\delta\right)\right)$. Using the formulation in proposition 1 and substituting (SE) into the problem, one gets the following expression for V_h:

$$V_h\left(w\right)=E_h\left(Y\right)-h\left(w+\underline{K}-\delta\underline{w}^*\right)+(1-p)\,\delta V\left(\underline{w}^*\right) \tag{9.1}$$

in which $\underline{w}^*$ solves the problem in proposition 1. The last term on the right-hand side $\left((1-p)\,\delta V\left(\underline{w}^*\right)\right)$ may be conditional on e_h and e_l. In order to prove that the principal eventually chooses e_l, assume by contradiction that high effort is always followed by high effort. Then one must have:

$$E_h\left(Y\right)-h\left(w+\underline{K}-\delta\underline{w}\right)+(1-p)\,\delta V_h\left(\underline{w}\right)\geq \tag{9.2}$$

$$E_h(Y) - h(w + \underline{K} - \delta \underline{w}) + (1-p)\delta V_h(\underline{w})$$

Which holds if and only if $V_h(\underline{w}) \geq V_l(\underline{w})$. If this inequality always holds, then it follows from proposition 5 that continuation values must decrease on average, which entails that $\{V'(w_t)\}_t$ is a non-positive submartingale. Then w must converge to a value $\hat{w} \geq w^* \geq u_{res}$. In the best scenario for the principal, it converges to u_{res}. Asymptotically, the principal stops using continuation values to provide incentives, which means that he offers the static contract that gives the agent a discounted utility u_{res}. But assumption, however, the principal would be strictly better off inducing low effort since $V(e_h/u_{res}) < V(e_l/u_{res})$. It follows that equation 9.2 cannot hold for every continuation value: there is one last period (in finite time) in which the principal chooses e_h.

Since the contract that induces low effort is stationary, w^* is the cutoff: the principal chooses e_h for $w > w^*$ and eventually turns to e_l at w^*.

□

Proof. Proposition 7

It suffices to show that the value function becomes linear as the agent's utility funcion becomes linear. Due to the envelope theorem, the derivative of the value function is:

$$V'(w) = -\frac{1 + \gamma\left[u'(\underline{c}) - u'(\overline{c})\right]}{pu'(\overline{c}) + (1-p)u'(\underline{c})}$$

In the limit when u becomes linear, the difference $[u'(\underline{c}) - u'(\overline{c})]$ goes to zero and since $\gamma = V'(\underline{w}) - V'(w)$ is bounded, the numerator converges to a constant. The denominator converges to the expected value of marginal utility (at the first-best), which is also constant. Hence V' converges to the inverse of the agent's marginal utility. Using the formulation of the problem in the proof of proposition 1, first-order conditions are (let ξ be the multiplier of the additional constraint $\overline{w} \geq u_{res}$):

$$(p+\lambda)\left[h'\left(w + \overline{K} - \delta\overline{w}\right) + V'(\overline{w})\right] + \xi = 0$$
$$(1-p)\left[h'(w + \underline{K} - \delta\underline{w}) + V'(\underline{w})\right] - \lambda h'(w + \underline{K} - \delta\underline{w}) + \eta = 0$$

The envelope theorem states that $V'(w) = -1$. Linearity implies $h'\left(w + \overline{K} - \delta\overline{w}\right) = h'\left(w + \overline{K} - \delta\overline{w}\right) = 1$ and $V'(\overline{w}) = V'(\underline{w}) = V'(w) = -1$. The first equation implies and $\xi = 0$, while the second entails $\lambda = \eta$; hence if the self-enforceability constraint is binding (as assumed), the limited liability constraint will always bind for the low continuation value w_l. As soon as the low outcome arises, the contract makes $w = u_{res}$ and amnesia arises: the contract is determined solely by the four constraints ((PK) will not actually depend on w as $w = u_{res}$). Lastly, notice that the principal will always choose high effort as there is no incentive benefit from low effort in a stationary contract.

□

Proof. Proposition 8

If $\eta = 0$, the proof of proposition 5 holds unchanged. If $\eta > 0$, it follows from this same proof that $E(V') = V' - \lambda\overline{V}' - \eta$, so that $E(V') < V'$ if η is high enough. This must necessarily be the case for w close enough to u_{res} as the principal cannot induce positive effort in the stage game: hence $\eta > -\lambda\overline{V}'$. To see that (LL) must bind eventually, notice that otherwise the principal would be using a higher lower bound for w, with no other change in the optimal contract; profits would increase strictly if he lets w decrease toward u_{res}.

For the behavior of effort, notice initially that it is a continuous function of w as the maximum theorem applies. The difference $u(c_h) - u(c_l)$ is decreasing in w due to (SE); applying the implicit function theorem to the first-order conditions of the principal's program, one has $\frac{\partial(w_h - w_l)}{\partial w} \to 0$ when w increases towards the upper bound $\frac{E\left(u(Y)/e^A\right) - K\left(e^A\right)}{1-\delta}$ (in which e^A solves$\max_e E[u(Y)/e] - K(e)$). Moreover, effort is increasing in $u(c_h) - u(c_l) + \beta(w_h - w_l)$ due to the agent's first-order condition. It follows that effort is decreasing in w; since w decreases over time before (LL) binds, effort increases. When w approaches the lower bound u_{res}, effort must start to decrease as it is continuous in w and must be equal to zero in the stage game.

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