Theory of the Firm

Based on Alexander Wolitzky (MIT – Microeconomics 1) and Leonardo Felli (LSE – Advanced Microeconomics)

Neoclassical Producer Theory in One Sentence

"Producers are just like consumers, but they maximize profit instead of utility."

We expand on this just slightly, and show how main results of producer theory follow from results from consumer theory.

The Profit Maximization Problem (PMP)

Choose production plan $y \in \mathbb{R}^n$ from production possibilities set $Y \subseteq \mathbb{R}^n$ to maximize profit $p \cdot y$:

$$\max_{y \in Y} p \cdot y$$

- Some prices can be negative.

Lets us model inputs and outpus symmetrically.

- Inputs have negative prices (firm pays to use them).
- Outputs have positive prices (firm makes money by producing).
- Neoclassical firm is price taker.
- No market power.
- Study of firms with market power is a topic in industrial organization.
- Firm's objective is profit maximization.
- In reality, firm is organization composed of individuals with different goals.
- Study of internal behavior and organization of firms is topic in organizational economics.

The PMP and the EMP

For our purposes, producer theory leaves everything interesting about firm behavior to other areas of economics, and reduces firm's problem to something isomorphic to consumer's expenditure minimization problem.

PMP is

$$\max_{y \in Y} \; p \cdot y.$$

Letting $S = \{x \in \mathbb{R}^n : u(x) \geq u\}$, EMP is

$$\min_{x \in S} p \cdot x.$$

Up to flipping a sign, PMP the same as EMP.

EMP: consumer chooses bundle of goods x to minimize expenditure, subject to x lying in set S.

PMP: firm chooses bundle of goods y to minimize net expenditure (maximize net profit), subject to y lying in set Y.

The PMP and the EMP

Solution to EMP: Hicksian demand $h(p)$.

Value function for EMP: expenditure function $e(p)$.

(omitting u because we hold it fixed)

Solution to PMP: optimal production plan $y(p)$.

Value function for EMP: profit function $\pi(p)$.

Producer theory: recall facts about Hicksian demand and expenditure function

Just translate into language of optimal production plan and profit function.

Properties of Hicksian Demand/Optimal Production Plans

Theorem

Hicksian demand satisfies:

1. Homogeneity of degree 0: for all $\lambda > 0, h(\lambda p) = h(p)$.
2. Convexity: if S is convex (i.e., if preferences are convex), then $h(p)$ is a convex set. (Singleton if strictly convex S.)
3. Law of demand: for every $p, p' \in \mathbb{R}^n, x \in h(p)$, and $x' \in h(p')$, we have $(p' - p)(x' - x) \leq 0$. (No income effect implies no Giffen good for firms.)

Theorem

Optimal production plans satisfy:

1. Homogeneity of degree 0 : for all $\lambda > 0, y(\lambda p) = y(p)$.
2. Convexity: if Y is convex, then $y(p)$ is a convex set. (Singleton if strictly convex Y)
3. Law of supply: for every $p, p' \in \mathbb{R}^n, y \in y(p)$, and $y' \in y(p')$, we have $(p' - p)(y' - y) \geq 0$.

Properties of Expenditure Function/Profit Function

Theorem

The expenditure function satisfies:

1. Homogeneity of degree 1: for all $\lambda > 0, e(\lambda p) = \lambda e(p)$.
2. Monotonicity: e is non-decreasing in p.
3. Concavity: e is concave in p.
4. Shephard's lemma: under mild conditions, e is differentiable, and $\frac{\partial}{\partial p_i} e(p) = h_i(p)$.

Theorem

The profit function satisfies:

1. Homogeneity of degree 1: for all $\lambda > 0, \pi(\lambda p) = \lambda \pi(p)$.
2. Monotonicity: π is non-decreasing in p.
3. Convexity: π is convex in p.
4. Hotelling's lemma: under mild conditions, π is differentiable, and $\frac{\partial}{\partial p_i} \pi(p) = y_i(p)$.

Exercise: use the information you have about the Slutsky matrix to conclude that output is increasing in price of output, and decreasing in price of input; and any input is decreasing in the price of that input. (This is just the law of demand we saw above.) What else does the Slutsky matrix tell us about the relationship between inputs and prices of inputs?

Production Plan

- Both outputs and inputs (measured in terms of flow) are services and commodities.
- y_j^o = quantity of commodity j produced by the firm as output,
- y_j^i = quantity of commodity j used as input,
- $z_j = y_j^o - y_j^i$ net output/input depending on whether the sign of z_j is positive/negative.
- Production plan = vector of net outputs and/or inputs of all available commodities

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_L \end{pmatrix}$$

Net Inputs and Outputs

- Without loss of generality we assume that:
- the first h commodities are net inputs
- while the remaining $L - h$ commodities are net outputs.
- Define:

$$x_1 = -z_1, \dots, x_h = -z_h, y_1 = z_{h+1}, \dots, y_{L-h} = z_L$$

Production Plan and Possibility Set

- A production plan is:

$$z = \begin{pmatrix} -x_1 \\ \vdots \\ -x_h \\ y_1 \\ \vdots \\ y_{L-h} \end{pmatrix}$$

Definition

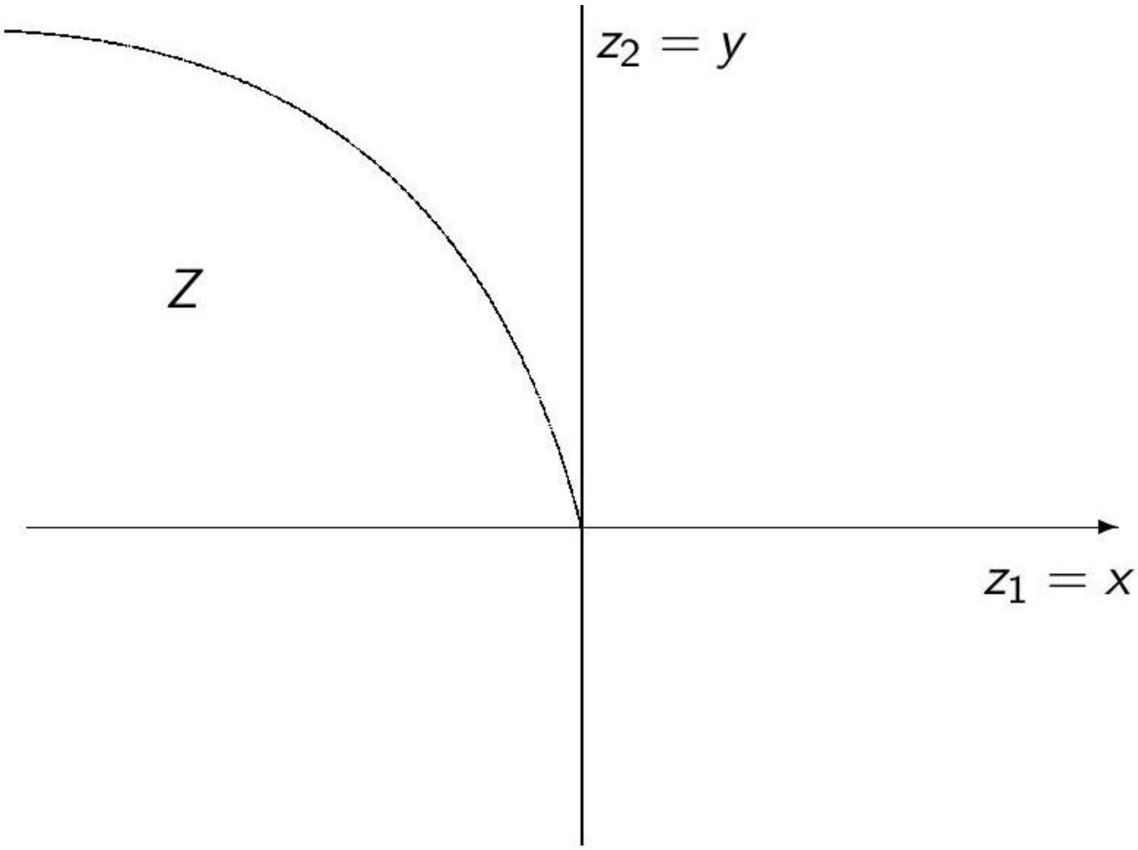
The Production Possibility Set $Z \subset \mathbb{R}^L$ (PPS) is the set of all technologically feasible production plans:

Set of all vectors of inputs and outputs that are technologically feasible.

PPS Z provides a complete description of the technology identified with the firm.

One Input One Output

Example of one input x and one output y production plan $z = \binom{-x}{y}$



Short-run PPS

- Sometime is possible to distinguish between:
- immediately technologically feasible production plans $Z(\bar{x}_1, \dots, \bar{x}_h)$;
- and eventually technologically feasible production plans Z.
- Consider

$$Z(\bar{x}_1) = \left\{ z = \begin{pmatrix} -x_1 \\ -x_2 \\ y \end{pmatrix} \mid x_1 = \bar{x}_1 \right\}$$

- For example if the input x_1 is fixed at the level $\bar{x}_1$ then we can define a short-run or restricted production possibility set.

Input Requirement Set and Isoquant

- A special feature of a technology is the input requirement set:

$$V(y) = \left\{x \in \mathbb{R}_+^h \mid \binom{-x}{y} \in Z\right\}$$

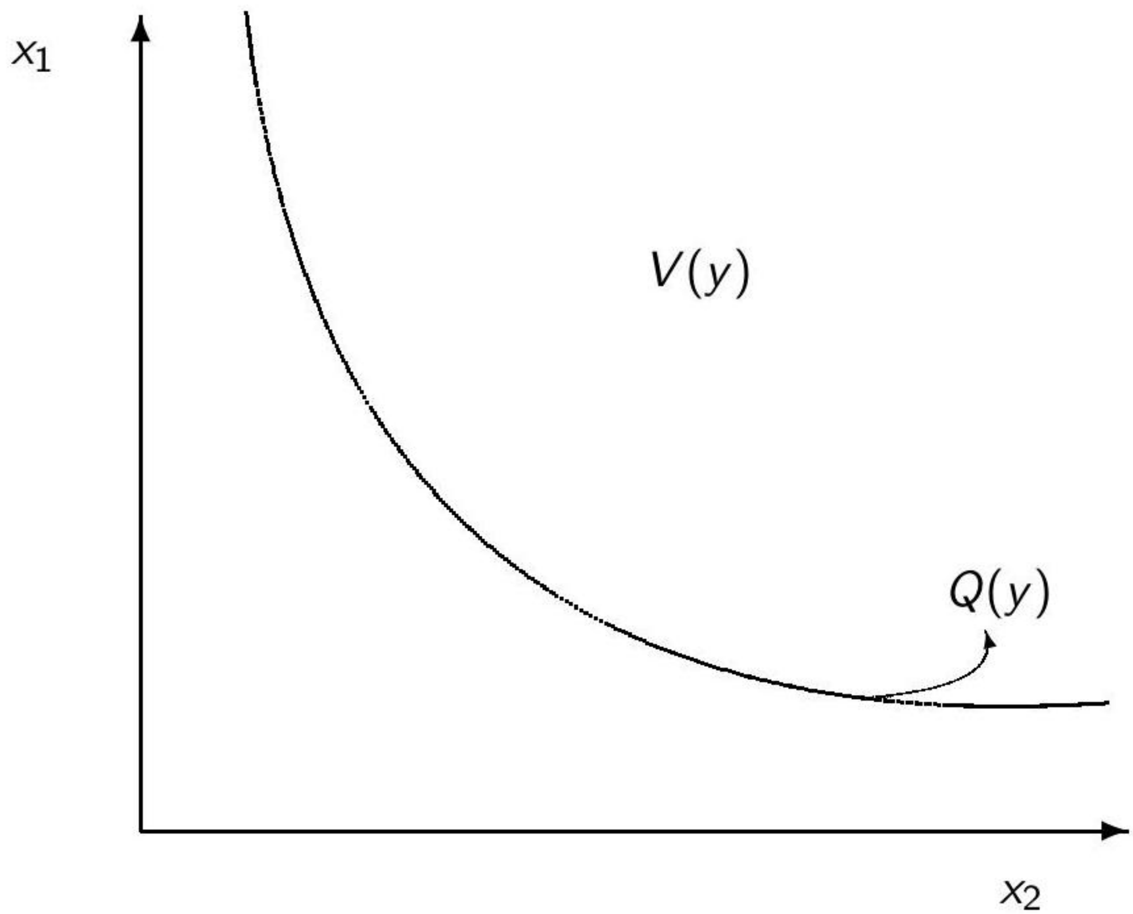
the set of all input bundles that produce at least y units of output.

- We define also the isoquant to be the set:

$$Q(y) = \{x \in \mathbb{R}_+^h \mid x \in V(y) \text{ and } x \notin V(y'), \forall y' > y\}$$

all input bundles that allow the firm to produce exactly y.

Two Inputs One Output



Technologically Efficient

In general, we can define the technologically efficient production plan z as:

Definition

The general production plan $z = \begin{pmatrix} -x \\ y \end{pmatrix} \in Z$ is technologically efficient, if and only if there does not exist a production plan $z' = \begin{pmatrix} -x' \\ y' \end{pmatrix} \in Z$ such that $z' \geq z (z_i' \geq z_i \forall i)$ and $z' \neq z$.

If z is efficient it is not possible to produce more output with a given input or the same output with less input (sign convention).

Production Function

Consider a technology with only one output

Definition

Production function in the case of only one output:

$$f(x) = \sup_{y'} \left\{ \begin{pmatrix} -x \\ y' \end{pmatrix} \in Z \right\}$$

the maximal output associated with the input bundle x.

Some Definitions

We can now introduce few definitions:

Definition

The <u>marginal product</u> of input x_i is

$$MP = \frac{\partial f(x)}{\partial x_i}$$

Definition

The <u>average product</u> of input x_i is

$$AP = \frac{f(x)}{x_i}$$

Definition

The <u>marginal rate of technical substitution</u> between input x_i and x_j is

$$MRTS = \left| \frac{dx_i}{dx_j} \right| = \frac{\partial f(x) / \partial x_j}{\partial f(x) / \partial x_i}$$

this is the absolute value of the slope of the isoquant.

The set of output bundles that are efficient for a given technology:

Definition

The Production Possibility Frontier:

$$\text{PPF}\,(x) = \left\{ y \mid \nexists z' \in Z \text{ s.t. } z' \geq z = \begin{pmatrix} -x \\ y \end{pmatrix} \right\}$$

Definition

The Marginal Rate of Transformation between output y_m and y_n is

$$MRT = \frac{dy_m}{dy_n}$$

as the slope of the PPF.

Definition

The elasticity of substitution between inputs i and j at some vector of inputs x^0 is:

$$\sigma_{ij}(x^0) = \left(\frac{d \ln MRTS \left(\frac{x_j}{x_i} \right)}{d\ ln \left(\frac{x_j}{x_i} \right)} \right)^{-1}$$

Percentage change in input proportion $\frac{x_j}{x_i}$ when MRTS changes by 1%.

Measure of curvature of isoquant.

Thae larger it is, the easier it is to substitute one input for another keeping production unchanged.

Example: Cobb-Douglas Technology

- We definite the Cobb-Douglas technology as

$$f(x_1, x_2) = x_1^{\alpha} x_2^{\beta},\ \alpha > 0, \beta > 0$$

or

$$Z = \left\{ \begin{pmatrix} -x_1 \\ -x_2 \\ y \end{pmatrix} \mid y \leq x_1^{\alpha} x_2^{\beta} \right\}$$

- with isoquant:

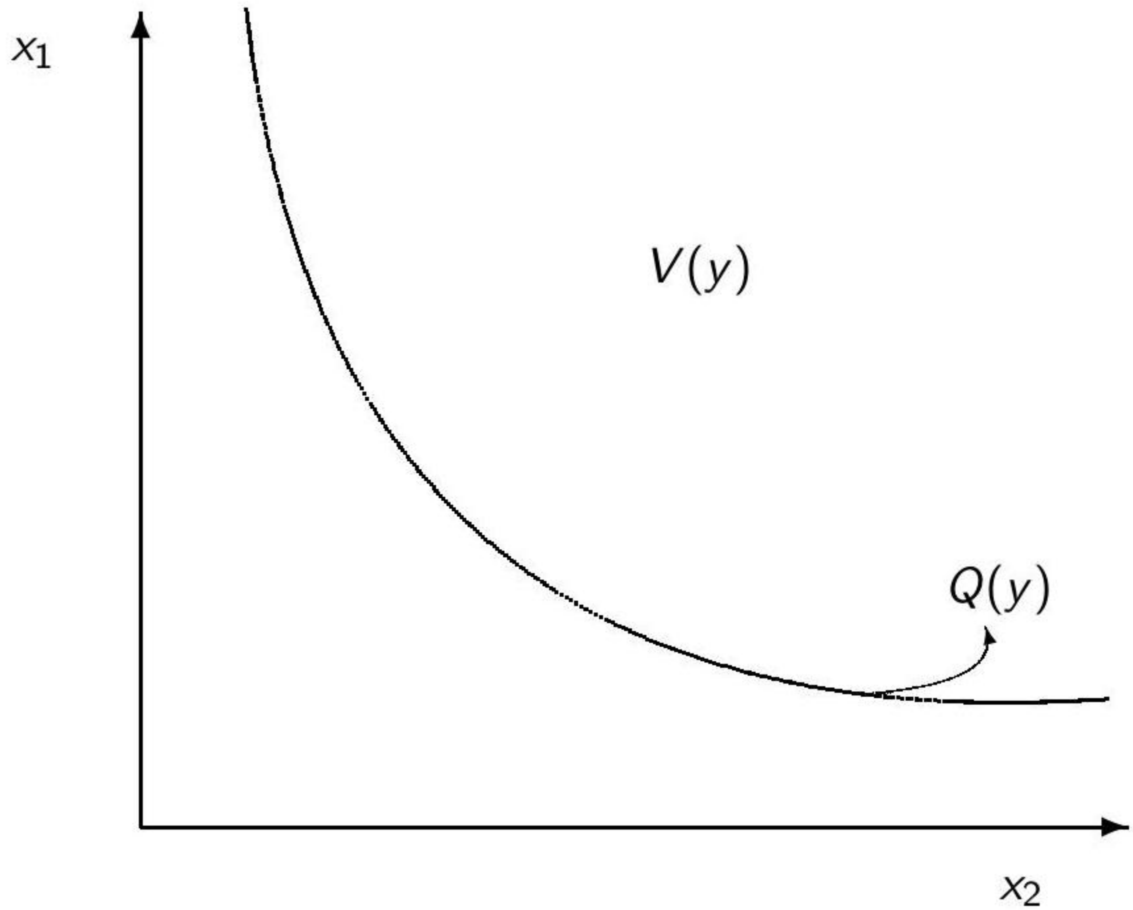
$$Q(y) = \left\{ (x_1, x_2) \in \mathbb{R}_+^2 \mid y = x_1^{\alpha} x_2^{\beta} \right\}$$

- and input requirement set:

$$V(y) = \left\{ (x_1, x_2) \in \mathbb{R}_+^2 \mid y \leq x_1^{\alpha} x_2^{\beta} \right\}$$

Elasticity of substitution is equal to one. (Check it here.)

Cobb-Douglas Isoquant



Example: Leontief Technology

- We definite the Leontief technology as

$$f(x_1, x_2) = \min\{ax_1, bx_2\},\ a > 0, b > 0$$

or

$$Z = \left\{ \begin{pmatrix} -x_1 \\ -x_2 \\ y \end{pmatrix} \mid y \leq \min\{ax_1, bx_2\} \right\}$$

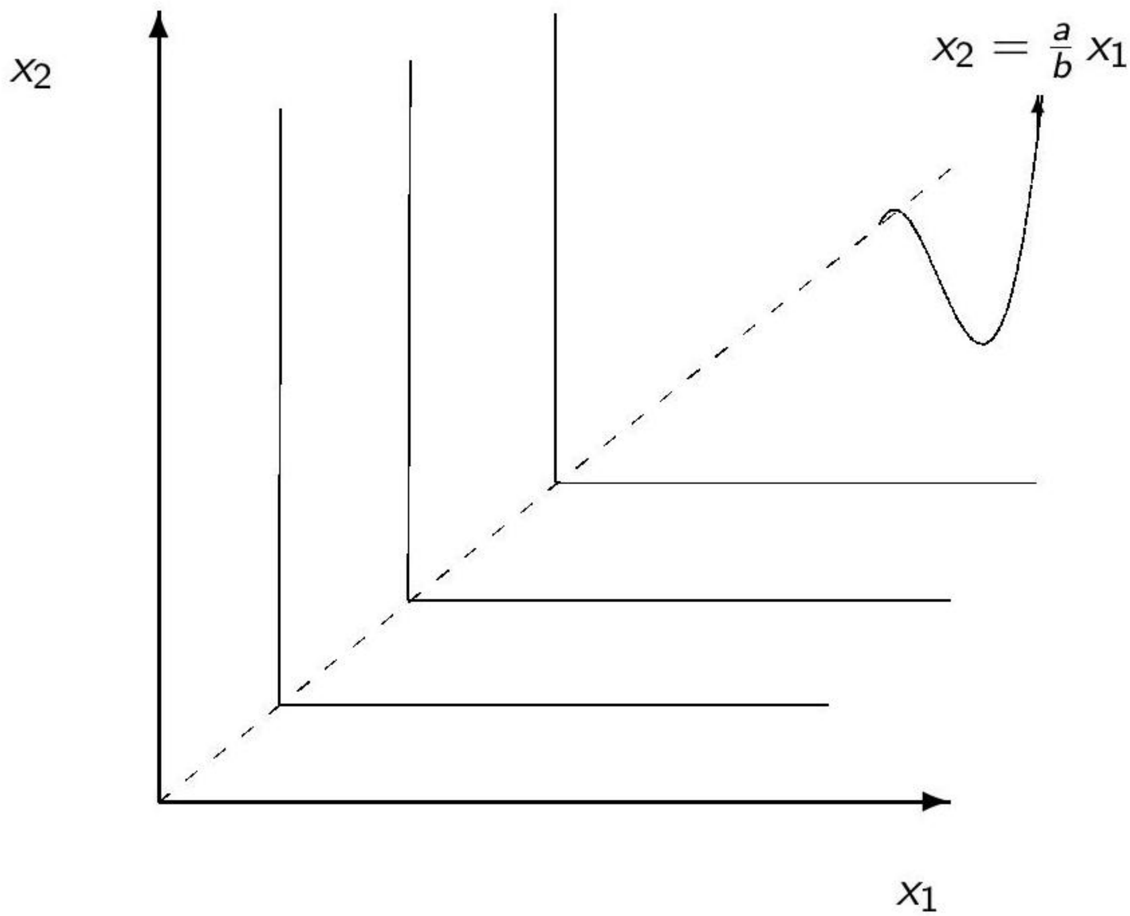
- with isoquant:

$$Q(y) = \left\{ (x_1, x_2) \in \mathbb{R}_+^2 \mid y = \min\{ax_1, bx_2\} \right\}$$

- and input requirement set:

$$V(y) = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid y \leq \min\{ax_1, bx_2\}\}$$

- where efficiency imposes $x_1 = \frac{y}{a}$, $x_2 = \frac{y}{b}$ Leontief Isoquants
- Elasticity of substitution is zero.



Example: Perfect Substitutes

- We definite the technology where inputs are perfect substitutes as

$$f(x_1, x_2) = ax_1 + bx_2,\ a > 0, b > 0$$

or

$$Z = \left\{ \begin{pmatrix} -x_1 \\ -x_2 \\ y \end{pmatrix} \mid y \leq ax_1 + bx_2 \right\}$$

- with isoquant:

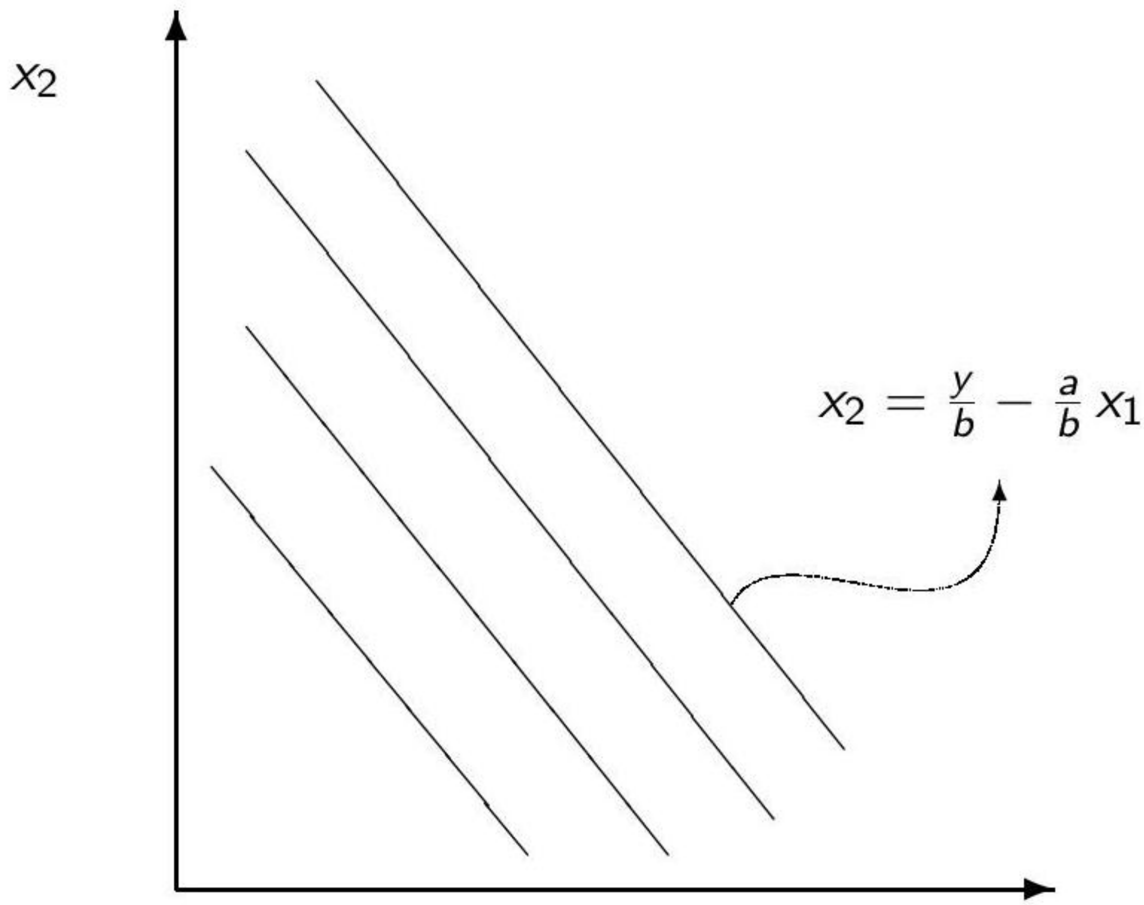
$$Q(y) = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid y = ax_1 + bx_2\}$$

- and input requirement set:

$$V(y) = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid y \le ax_1 + bx_2\}$$

Elasticity of substitution is infinite.

Perfect Substitutes Isoquants



CES Production Function

Former examples are particular cases of CES production function:

$$y = \left(x_1^\rho + x_2^\rho\right)^{\frac{1}{\rho}}$$

$$MRST = \left(\frac{x_2}{x_1}\right)^{1-\rho}$$

Elasticity of substitution:

$$\sigma = \frac{1}{1-\rho}$$

Check here the derivation, with a slightly different notation.

$\rho \longrightarrow 0 \Longrightarrow \sigma \longrightarrow 1$: Cobb-Douglas

$\rho \longrightarrow 1 \Longrightarrow \sigma \longrightarrow \infty$: perfect substitutes

$\rho \longrightarrow -\infty \Longrightarrow \sigma \longrightarrow 0$: perfect complements

Assumptions on PPS

Common assumptions on the PPS are:

1. **Z is closed (it contains its boundaries)**

   Important for the definition of a production function: implies sup is a max.

2. **$0 \in Z$: shut-down property**

   Uncontroversial property in the long run, not necessarily in the short run (inputs used with no outputs).

3. **Free disposal, monotonicity**

   If $z \in Z$ and $z' \leq z$ then $z' \in Z$.

   Alternatively: if $x \in V(y)$ and $x' \geq x$ then $x' \in V(y)$.

Given a feasible production plan if either one increases the quantity of inputs or reduces the quantity of output the new production plan is still feasible.

4. **Additivity**

   If $z, z' \in Z$ then $z + z' \in Z$ (stronger condition).

   For $f(x)$, this implies $f(x^1 + x^2) \geq f(x^1) + f(x^2)$.

5. **Convexity of $V(y)$**

   If $x, x' \in V(y)$ then $tx + (1-t)x' \in V(y)$ for every $0 \leq t \leq 1$

   $V(y)$ is convex set

6. **Convexity of Z**

   If $z, z' \in Z$ then $tz + (1-t)z' \in Z$ for every $0 \leq t \leq 1$, or Z is a convex set.

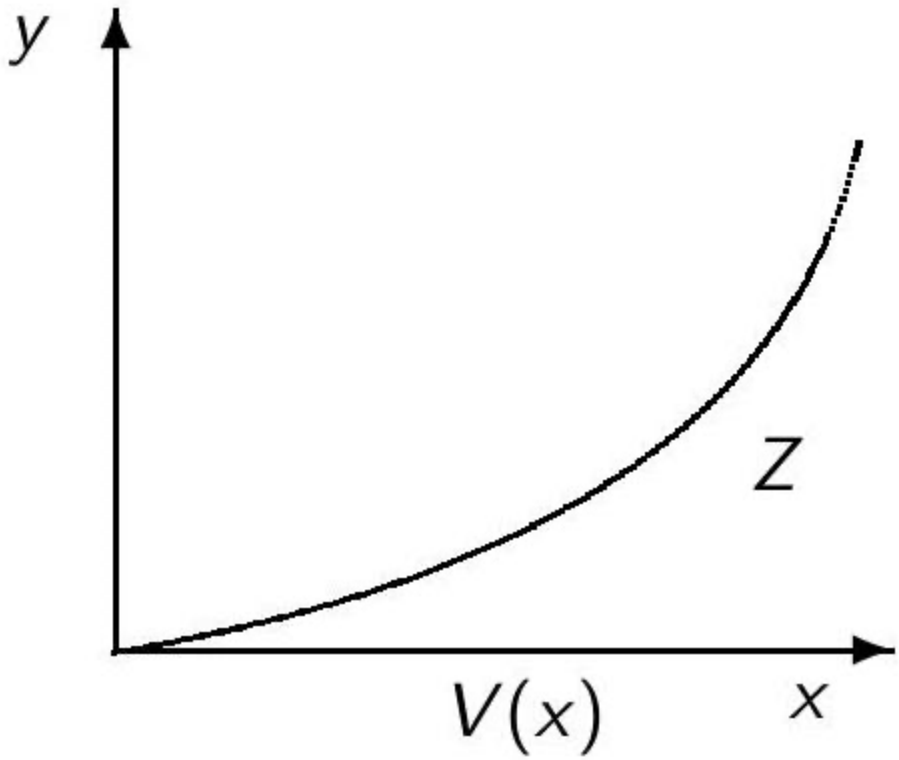
Notice that the (6) is stronger than (5): (6) $\Longrightarrow$ (5).

Some Results

Result

The convexity of Z implies the convexity of $V(y)$. The opposite implication does not hold.

Proof: It follows from the convexity of Z, the definition of $V(y)$ and the following counter-example of a one-input x and one output y technology.



Result

The convexity of $V(y)$ implies that the $f(x)$ is quasi-concave.

Proof: It follows from the convexity of $V(y)$ and the definition of a quasi-concave $f(x)$.

Definition

The function $f(\cdot)$ is quasi-concave if and only if the set $\{x \mid f(x) \geq k\}$ is convex for every $k \in \mathbb{R}$.

Notice that if you choose $k = y$, this set is $V(y)$.

Result

The convexity of Z implies that $f(x)$ is (weakly) concave.

Proof: Consider

$$z = \begin{pmatrix} -x \\ f(x) \end{pmatrix} \in Z,\ z' = \begin{pmatrix} -x' \\ f(x') \end{pmatrix} \in Z$$

Convexity of Z implies that for every $0 \leq t \leq 1$

$$tz + (1-t)z' = \begin{pmatrix} -(tx + (1-t)x') \\ tf(x) + (1-t)f(x') \end{pmatrix} \in Z$$

By definition of $f(x)$ this means:

$$tf(x) + (1-t)f(x') \leq f(tx + (1-t)x')$$

for every $0 \leq t \leq 1$. This is the definition of a concave $f(x)$.

Returns to Scale

- Decreasing Returns to Scale: (DRS) if $z \in Z$ then $tz \in Z$ for every $0 \leq t \leq 1$ (graph.).
- Increasing Returns to Scale: (IRS) if $z \in Z$ then $tz \in Z$ for every $t \geq 1$ (graph.).
- Constant Returns to Scale: (CRS) if $z \in Z$ then $tz \in Z$ for every $t \geq 0$ (graph.).

More Results

Result

Assumptions $0 \in Z$ and Z convex imply DRS.

Proof: It follows from the definition of convexity applied at $z' = 0$.

Result

A technology exhibits CRS if and only if the production function $f(x)$ (if available) is homogeneous of degree 1.

Proof: Assume CRS. This implies that if $z \in Z$ then $tz \in Z$, for all $t \geq 0$.

By definition of $Z, z \in Z$ means

$$y \leq f(x)$$

further $tz \in Z$ means

$$ty \leq f(tx).$$

By definition of production function choose z, and hence x and y, so that $y = f(x)$.

We can re-write the latter condition as:

$$tf(x) \leq f(tx)$$

We need to prove that the equality holds.

Suppose it does not. Then there exists y' such that

$$tf(x) < y' < f(tx)$$

Now $y' < f(tx)$ implies by definition of Z that

$$\begin{pmatrix} -tx \\ y' \end{pmatrix} \in Z$$

By CRS we get

$$\frac{1}{t}\begin{pmatrix} -tx \\ y' \end{pmatrix} \in Z \text{ or } \begin{pmatrix} -x \\ \frac{1}{t}y' \end{pmatrix} \in Z$$

which means

$$(1/t)y' \le f(x)$$

or

$$y' \le tf(x)$$

the latter inequality contradicts $tf(x) < y'$.

The opposite implication is an immediate consequence of the definition of homogeneity of degree 1 .

Conditions for DRS and IRS

Weaker conditions apply for DRS and IRS technology.

Result

Consider a technology characterized by a homogenous of degree $\alpha < 1(\alpha > 1)$ production function.

This technology exhibits DRS (IRS).

The opposite implication does not hold.

Result

Assume that $f(0) = 0$. Then:

- $f(x)$ concave implies DRS;
- $f(x)$ convex implies IRS;
- $f(x)$ concave and convex (that is, linear) implies CRS.

The Competitive Firm

- Assume that input and output prices are taken parametrically (no influence on such prices).
- As you have seen in consumer theory what this means is that whatever each firm decides in term of production does not affect the market.
- In other words, either firms are very small with respect to the market.
- Alternatively we are assuming that firms are not strategic: they do not realize that their choices trigger reactions in other firms in the market,
- or any of their potential choices would be taken into account by competitors when making their own choices.
- Additionally, we consider free entry (or perfect contestability), to economic profit is zero.
- There is no solution to the profit maximization problem for a competitive firm with increasing returns to scale!
- For constant returns to scale, optimal production is indeterminate.
- One may define local versions of returns to scale.
- Even if solution to profit maximization doesn't exist or is indeterminate, we can still learn from the firm's cost function.

Cost Function

Cost of output: expenditure it must make to acquire the inputs used to produce that output.

Technology will permit every level of output to be produced by difference input vectors.

These possibilities can be summarised by the level sets of the production function.

The firm must decide which of the possible production plans it will use.

Profit maximization implies cost minimization.

This holds for all firms, whether monopolists, perfect competitors, or anything between.

Output or input market may have some degree of market power from seller or buyer.

For now, keep assumption of perfect competition with input prices $\mathbf{w} = (w_1, \dots, w_n) \geq \mathbf{0}$ and inputs $\mathbf{x} = (x_1, \dots, x_n)$.

DEFINITION Cost Function

The cost function, defined for all input prices $\mathbf{w} \gg \mathbf{0}$ and all output levels $y \in f(\mathbb{R}^n_+)$, is the minimum-value function,

$$c(\mathbf{w}, y) \equiv \min_{\mathbf{x} \in \mathbb{R}^n_+} \mathbf{w} \cdot \mathbf{x} \;\text{ s.t. }\; f(\mathbf{x}) \geq y$$

If $\mathbf{x}(\mathbf{w}, y)$ solves the cost-minimisation problem, then

$$c(\mathbf{w}, y) = \mathbf{w} \cdot \mathbf{x}(\mathbf{w}, y)$$

constraint will always be binding at a solution if f is strictly increasing. Rewrite:

$$\min_{\mathbf{x} \in \mathbb{R}^n_+} \mathbf{w} \cdot \mathbf{x} \;\text{ s.t. }\; y = f(\mathbf{x}) \tag{3.1}$$

Let $\mathbf{x}^*$ denote a solution to (3.1).

Notice that this is similar to what we had initially: $Max\ p \cdot y$ subject to $y \in Y$

Previously, y denoted the whole vector of outputs and inputs

Now, it represents the output, and $x_1, \ldots, x_n$ are the inputs

We just have an additional restriction:

Not only "combination of outputs and inputs is feasible", but additionally $f(\mathbf{x}) \geq y$

This restriction does not affect the structure of the problem

Assume $\mathbf{x}^* \gg \mathbf{0}$, and that f is differentiable at $\mathbf{x}^*$ with $\nabla f(\mathbf{x}^*) \gg \mathbf{0}$. Hence there is a $\lambda^* \in \mathbb{R}$ such that

$$w_i = \lambda^* \frac{\partial f(\mathbf{x}^*)}{\partial x_i}, \; i = 1, \ldots, n$$

Because $w_i > 0, i = 1, \ldots, n$, we may divide the preceding i th equation by the j th to obtain

$$\frac{\partial f(\mathbf{x}^*)/\,\partial x_i}{\partial f(\mathbf{x}^*)/\,\partial x_j} = \frac{w_i}{w_j} \tag{3.2}$$

Cost minimisation implies that the marginal rate of substitution between any two inputs is equal to the ratio of their prices.

Solution depends on the parameters $\mathbf{w}$ and y.

Solution is unique if $\mathbf{w} \gg \mathbf{0}$ and f is strictly quasiconcave.

$\mathbf{x}^* \equiv \mathbf{x}(\mathbf{w}, y)$: vector of inputs minimising the cost of producing y units of output at the input prices $\mathbf{w}$: conditional input demand.

Cost function:

$$c(\mathbf{w}, y) = w_1 x_1(\mathbf{w}, y) + w_2 x_2(\mathbf{w}, y)$$

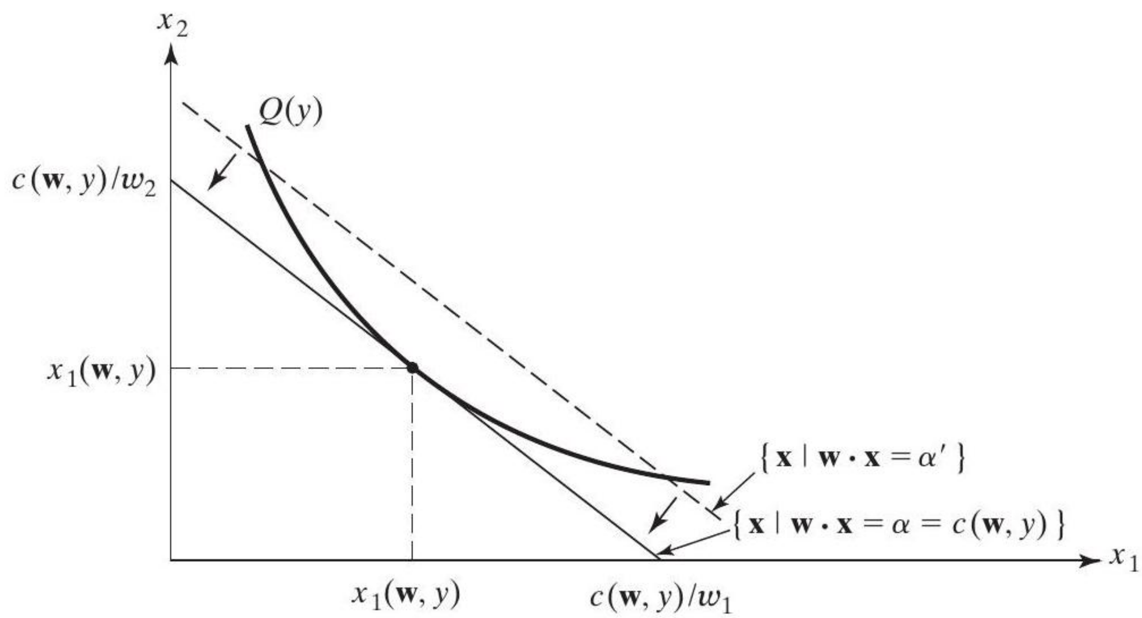


Figure 3.4. The solution to the firm's cost-minimisation problem.

EXAMPLE 3.3 CES production function.

$$\min_{x_1 \geq 0, x_2 \geq 0} w_1 x_1 + w_2 x_2 \text{ s.t. } \left(x_1^\rho + x_2^\rho\right)^{1/\rho} \geq y$$

Assuming $y > 0$ and an interior solution, the first-order Lagrangian conditions reduce to the two conditions

$$\frac{w_1}{w_2} = \left(\frac{x_1}{x_2}\right)^{\rho-1} \qquad (E.1)$$

$$y = \left(x_1^\rho + x_2^\rho\right)^{1/\rho} \qquad (E.2)$$

Solving (E.1) for x_1, substituting in (E.2), and rearranging gives

$$y = x_2 w_2^{-1/(\rho-1)} \left(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)}\right)^{1/\rho}$$

Solve this for x_2 and do the same for x_1 to obtain the conditional input demands:

$$x_1 = y w_1^{1/(\rho-1)} \left(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)}\right)^{-1/\rho} \qquad (E.3)$$

$$x_2 = y w_2^{1/(\rho-1)} \left(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)}\right)^{-1/\rho} \qquad (E.4)$$

Substitute (E.3) and (E.4) back into the objective function:

$$
\begin{aligned}
c(\mathbf{w}, y) &= w_1 x_1(\mathbf{w}, y) + w_2 x_2(\mathbf{w}, y) \\
&= y \left(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} \right)^{(\rho-1)/\rho}
\end{aligned}
$$

THEOREM 3.2 Properties of the Cost Function

1. If f is continuous and strictly increasing, then $c(\mathbf{w}, y)$ is
2. Zero when $y = 0$,
3. Continuous on its domain,
4. For all $\mathbf{w} \gg 0$, strictly increasing and unbounded above in y,
5. Increasing in $\mathbf{w}$,
6. Homogeneous of degree one in $\mathbf{w}$,
7. Concave in $\mathbf{w}$.

If f is strictly quasiconcave, then:

8. Shephard's lemma: $c(\mathbf{w}, y)$ is differentiable in $\mathbf{w}$ at $(\mathbf{w}^0, y^0)$ whenever $\mathbf{w}^0 \gg 0$, and

$$
\frac{\partial c(\mathbf{w}^0, y^0)}{\partial w_i} = x_i(\mathbf{w}^0, y^0),\ i = 1, \ldots, n
$$

EXAMPLE 3.4 Let $c(\mathbf{w}, y) = A w_1^{\alpha} w_2^{\beta} y$. From property 8 of Theorem 3.2, the conditional input demands are:

$$
x_1(\mathbf{w}, y) = \frac{\partial c(\mathbf{w}, y)}{\partial w_1} = \alpha A w_1^{\alpha-1} w_2^{\beta} y = \frac{\alpha c(\mathbf{w}, y)}{w_1} \qquad (E.1)
$$
$$
x_2(\mathbf{w}, y) = \frac{\partial c(\mathbf{w}, y)}{\partial w_2} = \beta A w_1^{\alpha} w_2^{\beta-1} y = \frac{\beta c(\mathbf{w}, y)}{w_2} \qquad (E.2)
$$

Ratio of conditional input demands:

$$
\frac{x_1(\mathbf{w}, y)}{x_2(\mathbf{w}, y)} = \frac{\alpha}{\beta} \frac{w_2}{w_1}
$$

It depends only on relative input prices, not on output.

Define input share:

$$
s_i \equiv w_i x_i(\mathbf{w}, y) / c(\mathbf{w}, y)
$$

From (E.1) and (E.2):

$$
\begin{aligned}
s_1 &= \alpha \\
s_2 &= \beta
\end{aligned}
$$

THEOREM 3.3 Properties of Conditional Input Demands

Assume f continuous, strictly increasing, strictly quasi-concave, $f(\mathbf{0}) = 0$. Assume cost function is twice continuously differentiable.

Then $\mathbf{x}(\mathbf{w}, y)$ is homogeneous of degree zero in $\mathbf{w}$,

The substitution matrix, defined and denoted

$$\boldsymbol{\sigma}^*(\mathbf{w}, y) \equiv \begin{pmatrix} \frac{\partial x_1(\mathbf{w}, y)}{\partial w_1} & \cdots & \frac{\partial x_1(\mathbf{w}, y)}{\partial w_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n(\mathbf{w}, y)}{\partial w_1} & \cdots & \frac{\partial x_n(\mathbf{w}, y)}{\partial w_n} \end{pmatrix}$$

is symmetric and negative semidefinite.

In particular, the negative semidefiniteness property implies that $\partial x_i(\mathbf{w}, y)/\, \partial w_i \leq 0$ for all i.

Homothetic Production Functions

Frequent in theoretical and applied work.

Homothetic production function: $f(x_1, x_2) = g[h(x_1, x_2)]$ for homogenous h and strictly increasing g.

THEOREM 3.4 Cost and Conditional Input Demands when Production is Homothetic

Assume f continuous, strictly increasing, strictly quasi-concave, homothetic, and $f(\mathbf{0}) = 0$. Then:

(a) the cost function is multiplicatively separable in input prices and output

It can be written $c(\mathbf{w}, y) = h(y)c(\mathbf{w}, 1)$
$h(y)$ is strictly increasing and $c(\mathbf{w}, 1)$ is the cost of 1 unit of output;

(b) the conditional input demands are multiplicatively separable in input prices and output

It can be written $\mathbf{x}(\mathbf{w}, y) = h(y)\mathbf{x}(\mathbf{w}, 1)$

$h'(y) > 0$ and $\mathbf{x}(\mathbf{w}, 1)$ is the conditional input demand for 1 unit of output.

When the production function is homogeneous of degree $\alpha > 0$,

(a) $c(\mathbf{w}, y) = y^{1/\alpha} c(\mathbf{w}, 1)$;

(b) $\mathbf{x}(\mathbf{w}, y) = y^{1/\alpha} \mathbf{x}(\mathbf{w}, 1)$.

Proof: Part 2 can be proved by mimicking the proof of part 1, so this is left as an exercise. Part 1(b) follows from Shephard's lemma, so we need only prove part 1(a).

Let F denote the production function. Because it is homothetic, it can be written as $F(\mathbf{x}) = f(g(\mathbf{x}))$, where f is strictly increasing, and g is homogeneous of degree one.

For simplicity, we shall assume that the image of F is all of $\mathbb{R}_+$. Consequently, as you are asked to show in Exercise 3.5, $f^{-1}(y) > 0$ for all $y > 0$. So, for some $y > 0$, let $t = f^{-1}(1)/f^{-1}(y) > 0$. Note then that $f(g(\mathbf{x})) \geq y \Leftrightarrow g(\mathbf{x}) \geq f^{-1}(y) \Leftrightarrow g(t\mathbf{x}) \geq tf^{-1}(y) = f^{-1}(1) \Leftrightarrow f(g(t\mathbf{x})) \geq 1$. Therefore, we may express the cost function associated with F as follows.

$$\begin{aligned} c(\mathbf{w}, y) &= \min_{\mathbf{x}\in\mathbb{R}^n} \mathbf{w} \cdot \mathbf{x} \text{ s.t. } f(g(\mathbf{x})) \geq y \\ &= \min_{\mathbf{x}\in\mathbb{R}^n_+} \mathbf{w} \cdot \mathbf{x} \text{ s.t. } f(g(t\mathbf{x})) \geq 1 \\ &= \frac{1}{t} \min_{\mathbf{x}\in\mathbb{R}^n} \mathbf{w} \cdot t\mathbf{x} \text{ s.t. } f(g(t\mathbf{x})) \geq 1 \\ &= \frac{1}{t} \min_{\mathbf{z}\in\mathbb{R}^n_+} \mathbf{w} \cdot \mathbf{z} \text{ s.t. } f(g(\mathbf{z})) \geq 1 \\ &= \frac{f^{-1}(y)}{f^{-1}(1)} c(\mathbf{w}, 1) \end{aligned}$$

where in the second to last line we let $\mathbf{z} \equiv t\mathbf{x}$.

Because f strictly increasing implies that f^{-1} is as well, the desired result holds for all $y > 0$. To see that it also holds for $y = 0$, recall that $c(\mathbf{w}, 0) = 0$, and note that $g(\mathbf{0}) = 0$, where the first equality follows from $F(\mathbf{0}) = 0$, and the second from the linear homogeneity of g.

QED.

Short run

So far, long run cost function: unrestricted choice of inputs.

Short run: some fixed inputs.

DEFINITION 3.6 The Short-Run, or Restricted, Cost Function

Let the production function be $f(\mathbf{z})$, where $\mathbf{z} \equiv (\mathbf{x}, \bar{\mathbf{x}})$ for a subvector of variable inputs $\mathbf{x}$ and a subvector of fixed inputs $\bar{\mathbf{x}}$.

$\mathbf{w}$ and $\bar{\mathbf{w}}$ be the associated input prices.

The short-run, or restricted, total cost function is defined as

$$sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}) \equiv \min_{\mathbf{x}} \mathbf{w} \cdot \mathbf{x} + \bar{\mathbf{w}} \cdot \bar{\mathbf{x}} \text{ s.t. } f(\mathbf{x}, \bar{\mathbf{x}}) \geq y$$

If $\mathbf{x}(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}})$ solves this minimisation problem, then

$$\text{sc}\,(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}) = \mathbf{w} \cdot \mathbf{x}(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}) + \bar{\mathbf{w}} \cdot \bar{\mathbf{x}}$$

Optimised cost of the variable inputs, $\mathbf{w} \cdot \mathbf{x}(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}})$: total variable cost.

Given cost of the fixed inputs: $\bar{\mathbf{w}} \cdot \bar{\mathbf{x}}$: total fixed cost.

Long run costs ≥ short-run costs

Any cost function achievable in short run is achievable in long run (more freedom)

Assume $w_1 = 1$: horizontal intercepts measure indicated costs. If in the short run, the firm is stuck with $\bar{x}_2$ units of the fixed input, it must use input combinations A, C, and E, to produce output levels y^1, y^2, and y^3, and incur short-run costs of $sc(y^1), sc(y^2)$, and $sc(y^3)$, respectively. In the long run, when the firm is free to choose

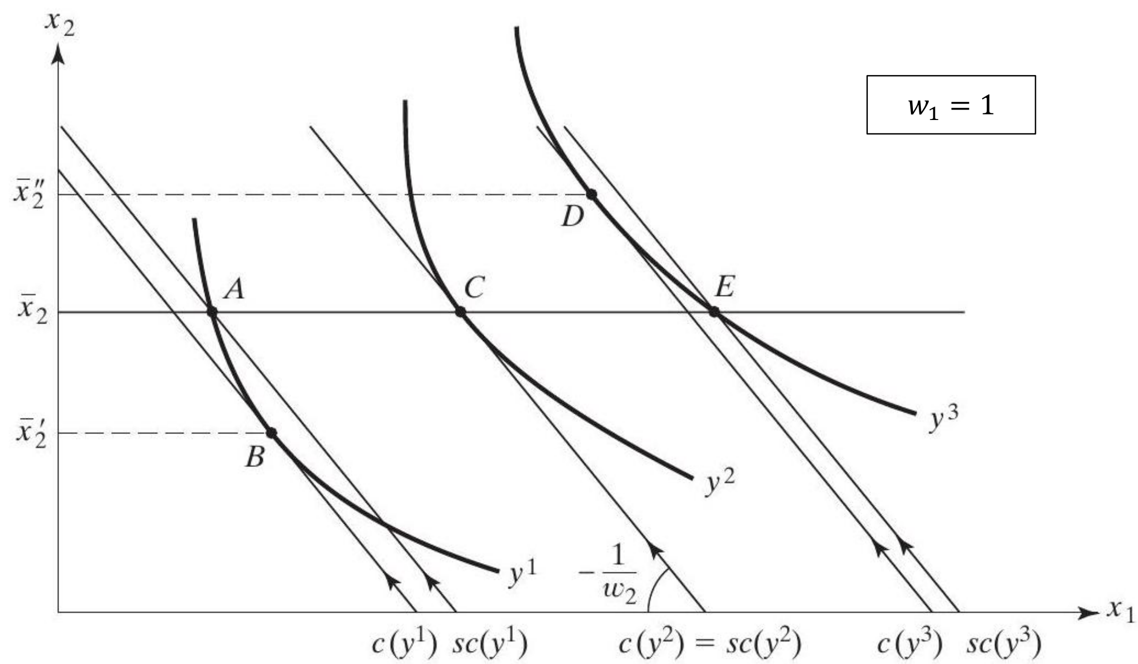


Figure 3.5. $sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}) \geq c(\mathbf{w}, \bar{\mathbf{w}}, y)$ for all output levels y.

both inputs optimally, it will use input combinations B, C, and D, and be able to achieve long-run costs of $c(y^1), c(y^2)$, and $c(y^3)$, respectively.

$\bar{x}_2$ is exactly the amount of x_2 the firm would choose to use in the long run to produce y^1 at the prevailing input prices.

Thus, there can be no difference between long-run and short-run costs at that level of output.

Each different level of the fixed input would give rise to a different shortrun cost function, yet in each case, short-run and long-run costs would coincide for some particular level of output.

Let $\bar{\mathbf{x}}(y)$ denote the optimal choice of the fixed inputs to minimise short-run cost of output y at the given input prices. Then **for all** y:

$$c(\mathbf{w}, \bar{\mathbf{w}}, y) \equiv \mathrm{sc}\,(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}(y)) \tag{3.3}$$

Fixed inputs chosen to minimise shortrun costs implies:

$$\frac{\partial sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}(y))}{\partial \bar{x}_i} \equiv 0 \tag{3.4}$$

Differentiate identity (3.3) and use (3.4):

$$\frac{dc(\mathbf{w}, \bar{\mathbf{w}}, y)}{dy} = \frac{\partial sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}(y))}{\partial y} + \underbrace{\sum_i \frac{\partial sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}(y))}{\partial \bar{x}_i} \frac{\partial \bar{x}_i(y)}{\partial y}}_{=0}$$

Hence:

$$\frac{dc(\mathbf{w}, \bar{\mathbf{w}}, y)}{dy} = \frac{\partial sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}(y))}{\partial y}$$

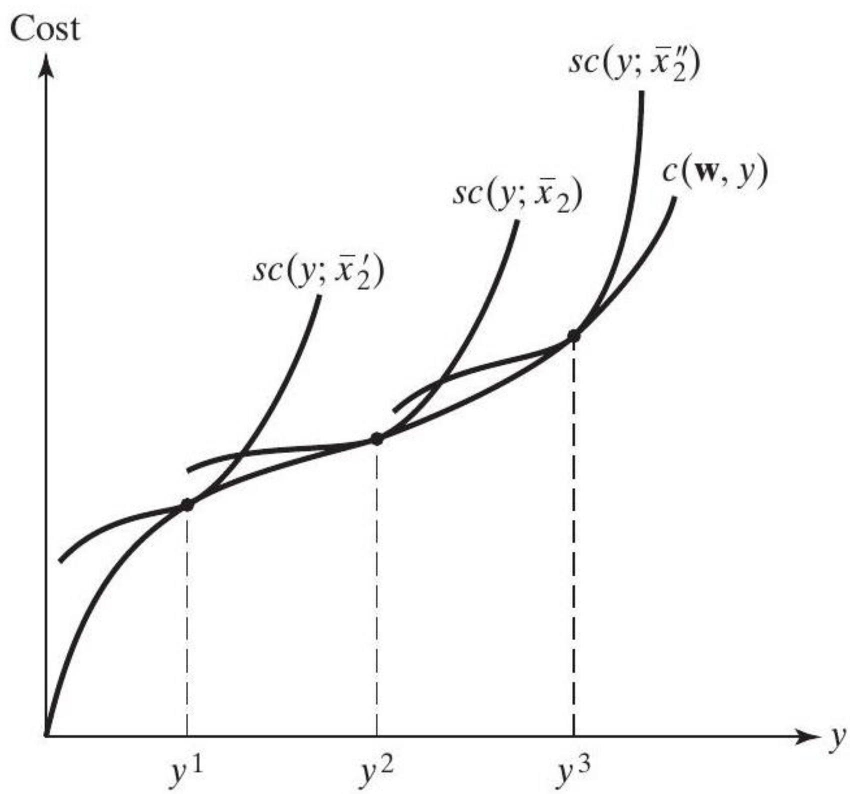
Summarizing:

$$sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}) \geq c(\mathbf{w}, \bar{\mathbf{w}}, y)$$

$$c(\mathbf{w}, \bar{\mathbf{w}}, y) \equiv \mathrm{sc}\,(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}(y))$$

$$\frac{dc(\mathbf{w}, \bar{\mathbf{w}}, y)}{dy} = \frac{\partial sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}(y))}{\partial y}$$

Long-run total cost curve is the lower envelope of the entire family of short-run total cost curves.



Exercise: derive the average cost function for (global) IRS, CRS and DRS.

Closer Look at Profit Maximization

The basic producer problem is than profit maximization:

$$\max_{\{x,y\}} py - \sum_{i=1}^{h} w_i x_i$$
$$\text{s.t. } \begin{pmatrix} -x \\ y \end{pmatrix} \in Z$$

where p and w_i are taken as parameters.

Let:

- the h-dimensional vector of input prices be $w = (w_1, \dots, w_h)$;
- the L - h-dimensional vector of output prices be $p = (p_1, \dots, p_{L-h})$.

We can re-write the producer's problem as:

$$\max_{\{z\}} \quad \hat{p}z$$
$$\text{s.t.} \quad z \in Z$$

where $\hat{p} = (w, p)$ and $z = \begin{pmatrix} -x \\ y \end{pmatrix}$.

This is what we've seen before, with a slightly different notation: z instead of y.

Profit Maximization with only one output: differentiable case

In the case of a technology that produces only one output the profit maximization problem may be written as:

$$\max_{\{x,y\}} pf(x) - wx$$

The necessary first order conditions of this problem are:

$$p\nabla f(x^*) \leq w$$

or

$$\frac{\partial f(x^*)}{\partial x_i} \leq \frac{w_i}{p}, \ \forall i = 1, \dots, h$$

and

$$\left[\frac{\partial f(x^*)}{\partial x_i} - \frac{w_i}{p}\right] x_i^* = 0, \ \forall i = 1, \dots, h.$$

Profit Maximization: Second-order conditions

In the event that the production possibility set is convex (the production function is concave) the first order conditions are both necessary and sufficient.

- In other case, the following set of sufficient conditions for a local maximum has to be verified.
- The Hessian matrix of the production function has to be negative definite at the point x^*.

This condition can be checked by the sufficient determinant condition according to which the leading principal minors have to alternate sign starting from the negative one.

For the case of two variables the first order conditions are for $i = 1,2$:

$$\frac{\partial f(x^*)}{\partial x_i} \leq \frac{w_i}{p}$$

and

$$\left[\frac{\partial f(x^*)}{\partial x_i} - \frac{w_i}{p}\right] x_i^* = 0$$

while the second order conditions are:

$$H = \begin{pmatrix} \frac{\partial^2 f(x^*)}{\partial x_1^2} & \frac{\partial^2 f(x^*)}{\partial x_1 \, \partial x_2} \\ \frac{\partial^2 f(x^*)}{\partial x_1 \, \partial x_2} & \frac{\partial^2 f(x^*)}{\partial x_2^2} \end{pmatrix} \text{ negative definite}$$

Which is implied by:

$$\frac{\partial^2 f(x^*)}{\partial x_i^2} < 0$$

and

$$\frac{\partial^2 f(x^*)}{\partial x_1^2} \frac{\partial^2 f(x^*)}{\partial x_2^2} - \left(\frac{\partial^2 f(x^*)}{\partial x_1 \, \partial x_2}\right)^2 > 0$$

Unconditional Factor Demand and Supply Function

The solution to the profit maximization problem if it exists provides the unconditional factor demands:

$$x(p, w) = x^*$$

By substitution it is possible to obtain the supply function of the producer:

$$y(p, w) = f(x(p, w)).$$

Comparative statics results obtained by differentiating the FOC (they are identities in (p, w)).

Properties of Factor Demands

1. Non-positive own factor demands price effects (SOC) (generalizes to h inputs):

$$\frac{\partial x_1}{\partial w_1} \leq 0$$

$$\frac{\partial x_2}{\partial w_2} \leq 0$$

2. Symmetry (generalizes to h inputs):

$$\frac{\partial x_1}{\partial w_2} = \frac{\partial x_2}{\partial w_1}$$

3. Complementary inputs (generalizes to h inputs):

$$\frac{\partial x_1}{\partial w_2} = \frac{\partial x_2}{\partial w_1} < 0$$

4. Substitutability of inputs (it does not generalize to several inputs):

$$\frac{\partial x_1}{\partial w_2} = \frac{\partial x_2}{\partial w_1} > 0$$

5. Finally positive output price effects (generalizes to h inputs):

$$\frac{\partial x_1}{\partial p} > 0 \; \frac{\partial x_2}{\partial p} > 0$$

(If x_1 and x_2 are complementary inputs.)

Some comparative static results obtained differentiating the supply function of the firm:

$$y(p, w) = f(x(p, w))$$

6. Own price effect non-negative:

$$\frac{\partial y}{\partial p} \geq 0$$

7. Symmetry:

$$-\frac{\partial x_i}{\partial p} = \frac{\partial y}{\partial w_i}$$

for $i = 1,2$.

Summary of the Properties

Such comparative statics properties can be summarized as:

$$\begin{pmatrix} \frac{\partial y}{\partial p} & \frac{\partial y}{\partial w_1} & \frac{\partial y}{\partial w_2} \\ -\frac{\partial x_1}{\partial p} & -\frac{\partial x_1}{\partial w_1} & -\frac{\partial x_1}{\partial w_2} \\ -\frac{\partial x_2}{\partial p} & -\frac{\partial x_2}{\partial w_1} & -\frac{\partial x_2}{\partial w_2} \end{pmatrix} \text{ s.t. } \begin{pmatrix} + & a & b \\ a & + & c \\ b & c & + \end{pmatrix}$$

Other Properties

8. Both $x(p,w)$ and $y(p,w)$ are homogeneous of degree 0 .

Proof: If you increase both input and output prices by a factor $t > 0$ you obtain:

$$\max_x (tp)f(x) - (tw)x = \max_x t[pf(x) - wx]$$

which clearly is solved by the same vector $x(p,w)$ that solves:

$$\max_x pf(x) - wx$$

Further, by definition of supply function:

$$y(tp,tw) = f(x(tp,tw)) = f(x(p,w)) = y(p,w) \;\square$$

Profit Function

Definition

The following is defined as the <u>profit function</u>

$$\pi(p,w) = \max_x pf(x) - wx = pf(x(p,w)) - wx(p,w)$$

Properties:

1. Price effects:

$$\frac{\partial \pi}{\partial w_i} \leq 0$$

$$\frac{\partial \pi}{\partial p} \geq 0$$

2. The profit function $\pi(p,w)$ is homogeneous of degree 1 in (p,w).

Proof: It follows from the homogeneity of degree 0 of $x(p,w)$ and, for any scalar α,

$$\begin{aligned}
\pi(\alpha p, \alpha w) \quad &= \alpha p f(x(\alpha p, \alpha w)) - \alpha w x(\alpha p, \alpha w) \\
&= \alpha p f(x(p,w)) - \alpha w x(p,w) \\
&= \alpha[pf(x(p,w)) - wx(p,w)] \\
&= \alpha \pi(p,w)
\end{aligned}$$

3. Hotelling Lemma (which proves property 1):

$$\frac{\partial \pi}{\partial p} = y(p,w) \geq 0 \;\text{ and }\; \frac{\partial \pi}{\partial w_i} = -x_i(p,w) \leq 0$$

Proof: It follows by Envelope Theorem applied to

$$\pi(p,w) = \max_x pf(x) - wx = pf(x(p,w)) - wx(p,w)$$

4. The profit function $\pi(p, w)$ is convex in (p, w).

Proof: Consider the two price vectors (p, w) and (p', w') and for every scalar $\lambda \in (0,1)$ let

$$p'' = \lambda p + (1 - \lambda)p'$$

and

$$w'' = \lambda w + (1 - \lambda)w'$$

- Then:

$$\begin{aligned}
\pi[p'', w''] = \quad & p'' f\big(x(p'', w'')\big) - w'' x(p'', w'') \\
= \quad & \lambda\big[p f\big(x(p'', w'')\big) - w x(p'', w'')\big] \\
& \quad + (1 - \lambda)\big[p' f\big(x(p'', w'')\big) - w' x(p'', w'')\big] \\
\leq \quad & \lambda \pi(p, w) + (1 - \lambda)\pi(p', w')
\end{aligned}$$

which proves convexity of $\pi(p, w)$.

Profit Maximization with Cost Function

Frequently we will write the firm's profit using the cost function:

$$py - c(y)$$

If c is differentiable, solution is:

$$p \leq c'(y)$$

With equality if $y > 0$.

Second-order condition: $0 \leq c''$: convex cost.

Minimum cost and competitive behavior

If there is some y^* that minimizes the average cost ${}^{c(y)}/_{y}$, one may show that

$$\frac{dc(y^*)}{dy} = \frac{c(y^*)}{y^*}$$

Marginal cost equals average cost

But we know that:

Profit maximization and price taking behavior imply $p = \frac{dc(y^*)}{dy}$

Free entry (or perfect contestability) implies $p = \frac{c(y^*)}{y}$

Hence profit maximization, price taking behavior and free entry imply:

$$\frac{dc(y^*)}{dy} = \frac{c(y^*)}{y}$$

That is, minimum average cost.

Remember the following are equivalent:

Price equal to average cost

No economic profit.

No extraordinary remuneration of the factors of production

Monotone Comparative Statics: Motivation

Comparative statics are statements about how solution to a problem changes with parameters.

Core of most applied economic analysis.

Last twenty years or so:

revolution in how comparative statics are done in economics.

Traditional approach: differentiate FOC using implicit function theorem.

New approach:

monotone comparative statics.

Example: Traditional Approach

Consider problem:

$$\max_{x \in X} b(x,\theta) - c(x)$$

- x is choice variable
- θ is parameter
- $b(x,\theta)$ is benefit from choosing x given parameter θ
- c is cost of choosing x

Example: Traditional Approach

$$\max_{x \in X} b(x,\theta) - c(x)$$

If $X \subseteq \mathbb{R}$ and b and c are differentiable, FOC is

$$b_x(x^*(\theta),\theta) = c'(x^*(\theta)).$$

If b and c are twice continuously differentiable and $b_{xx}(x^*(\theta),\theta) \neq c''(x^*(\theta))$, implicit function theorem implies that solution $x^*(\theta)$ is continuously differentiable, with derivative

$$\frac{d}{d\theta}x^*(\theta) = \frac{b_{x\theta}(x^*(\theta),\theta)}{c''(x^*(\theta)) - b_{xx}(x^*(\theta),\theta)}.$$

If c is convex, b is concave in x, and $b_{x\theta} > 0$, can conclude that $x^*(\theta)$ is (locally) increasing in θ. Intuition: FOC sets marginal benefit equal to marginal cost. If $b_{x\theta} > 0$ and θ increases, then if b is concave in x and c is convex, x must increase to keep the FOC satisfied.

What's Wrong with This Picture?

Unnecessary assumptions: as we'll see, solution(s) are increasing in θ even if b is not concave, c is not convex, b and c are not differentiable, and choice variable is not continuous or real-valued.

Wrong intuition: Intuition coming from the FOC involves concavity of b and convexity of c.

This can't be the right intuition.

We'll see that what's really needed is an ordinal condition on b-the single-crossing property-which is a more meaningful version of the assumption $b_{x\theta} > 0$.

Why Learn Monotone Comparative Statics?

Three reasons:

1. Generality: Cut unnecessary convexity and differentiability assumptions.

2. Analytical power: Often, can't assume convexity and differentiability.

(Traditional approach doesn't work.)

3. Understanding: By focusing on essential assumptions, help to understand workings of economic models.

(Don't get confused about what drives what.)

Fourth reason: need to understand them to read other people's papers.

- Costinot, A. "An Elementary Theory of Comparative Advantage." Econometrica, 2009. [International]

- Acemoglu, D. "When Does Labor Scarcity Encourage Innovation?" Journal of Political Economy, 2010. [Growth/Innovation]

- Kircher, P. and J. Eeckhout. "Sorting and Decentralized Price Competition." Econometrica, 2010. [Labor]

- Segal, I. and M. Whinston. "Property Rights." Chapter for Handbook of Organizational Economics, 2011.

[Organizational Econ]

- Acemoglu, D. and A. Wolitzky. "The Economics of Labor Coercion." Econometrica, 2011. [Political Economy]

MCS with 1 Choice Variable and 1 Parameter

Start with simple case: $X \subseteq \mathbb{R}, \Theta \subseteq \mathbb{R}$.

Interested in set of solutions $X^*(\theta)$ to optimization problem

$$\max_{x \in X} f(x, \theta)$$

Under what conditions on f is $X^*(\theta)$ increasing in θ ?

The Strong Set Order

What does it mean for set of solutions to be increasing?

Relevant order on sets: strong set order.

Definition

A set $A \subseteq \mathbb{R}$ is greater than a set $B \subseteq \mathbb{R}$ in the strong set order (SSO) if, for any $a \in A$ and $b \in B$,

$$\begin{aligned} \max\{a, b\} &\in A, \text{ and} \\ \min\{a, b\} &\in B. \end{aligned}$$

$X^*(\theta)$ greater than $X^*(\theta')$ if, whenever x is solution at θ and x' is solution at θ', either

1. $x \geq x'$, or
2. both x and x' are solutions for both parameters.

Increasing Differences

Simple condition on f that guarantees that $X^*(\theta)$ is increasing (in SSO) : increasing differences.

Definition

A function $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ has increasing differences in (x, θ) if, whenever $x^H \geq x^L$ and $\theta^H \geq \theta^L$, we have

$$f(x^H, \theta^H) - f(x^L, \theta^H) \geq f(x^H, \theta^L) - f(x^L, \theta^L).$$

Return to choosing a higher value of x is non-decreasing in θ.

Form of complementarity between x and θ.

Increasing Differences: Differential Version

Theorem

If f is twice continuously differentiable, then f has increasing differences in (x, θ) iff

$$\frac{\partial^2 f(x,\theta)}{\partial x \, \partial \theta} \geq 0 \text{ for all } x \in X, \theta \in \Theta.$$

Increasing differences generalizes condition on cross-partial derivatives used to sign comparative statics in traditional approach.

Topkis' Monotonicity Theorem

Simplest MCS theorem:

Theorem (Topkis)

If f has increasing differences in (x, θ), then $X^*(\theta)$ is increasing in the strong set order.

Back to Example

$$\max_{x \in X} b(x,\theta) - c(x)$$

If b has increasing differences in (x, θ), then $X^*(\theta)$ is increasing in the strong set order.

No assumptions about convexity or differentiability of anything.

Necessity

Want to find minimal assumptions for given comparative statics result to hold.

Is increasing differences minimal assumption?

No: increasing differences is cardinal property, but property that $X^*(\theta)$ is increasing is ordinal.

What's ordinal version of increasing differences?

Single-Crossing

Definition

A function $f: X \times \Theta \to \mathbb{R}$ is single-crossing in (x, θ) if, whenever $x^H \geq x^L$ and $\theta^H \geq \theta^L$, we have

$$f(x^H, \theta^L) \geq f(x^L, \theta^L) \Longrightarrow f(x^H, \theta^H) \geq f(x^L, \theta^H)$$

and

$$f(x^H, \theta^L) > f(x^L, \theta^L) \Longrightarrow f(x^H, \theta^H) > f(x^L, \theta^H).$$

Whenever choosing a higher x is better at a low value of θ, it's also better at a high value of θ.

Increasing differences implies single-crossing, but not vice versa.

Milgrom-Shannon Monotonicity Theorem

Theorem (Milgrom and Shannon)

If f is single-crossing in (x, θ), then $X^*(\theta)$ is increasing in the strong set order.

Conversely, if $X^*(\theta)$ is increasing in the strong set order for every choice set $X \subseteq \mathbb{R}$, then f is single-crossing in (x, θ).

Strictly Increasing Selections

A stronger set order: for $\theta < \theta'$, every $x \in X^*(\theta)$ is strictly less than every $x' \in X^*(\theta')$.

(Every selection is strictly increasing.)

When is every selection strictly increasing?

Strictly increasing differences: whenever $x^H > x^L$ and $\theta^H > \theta^L$, we have

$$f(x^H, \theta^H) - f(x^L, \theta^H) > f(x^H, \theta^L) - f(x^L, \theta^L).$$

Theorem (Edlin and Shannon)

Suppose f is continuously differentiable in x and has strictly increasing differences in (x, θ).

Then, for all $\theta < \theta', x^* \in X^*(\theta) \cap \text{int } X$, and $x^{*\prime} \in X^*(\theta')$, we have $x^* < x^{*\prime}$.

MCS with n Choice Variables and m Parameters

Previous theorems generalize to $X \subseteq \mathbb{R}^n$ and $\Theta \subseteq \mathbb{R}^m$.

Two main issues in generalization:

1. What's "max" or "min" of two vectors?

2. Need complementarity within components of x, not just between x and θ.

Once clear these up, analysis same as in 1-dimensional case.

Meet and Join

Relevant notion of min and max are component-wise min and max, also called meet and join:

$$x \wedge y = (\min\{x_1, y_1\}, \dots, \min\{x_n, y_n\})$$
$$x \vee y = (\max\{x_1, y_1\}, \dots, \max\{x_n, y_n\})$$

Definition

A set $A \subseteq \mathbb{R}^n$ is greater than a set $B \subseteq \mathbb{R}^n$ in the strong set order if, for any $a \in A$ and $b \in B$,

$$a \vee b \in A, \text{ and}$$
$$a \wedge b \in B.$$

A lattice is a set $X \subseteq \mathbb{R}^n$ such that $x \wedge y \in X$ and $x \vee y \in X$ for all $x, y \in X$.

Ex. A product set $X = X_1 \times \dots X_n$ is a lattice.

Increasing Differences

Definition of increasing differences in (x, θ) same as before: $x^H \geq x^L, \theta^H \geq \theta^L \Longrightarrow$

$$f(x^H, \theta^H) - f(x^L, \theta^H) \geq f(x^H, \theta^L) - f(x^L, \theta^L)$$

(Note: x and θ are vectors. What does $x^H \geq x^L$ mean?)

Increasing differences in (x, θ) no longer enough to guarantee $X^*(\theta)$ increasing.

Issue: what if increase in θ_1 pushes x_1 and x_2 up, but increase in x_1 pushes x_2 down?

Need complementarity within components of x, not just between x and θ.

27 This is called supermodularity of f in x.

Supermodularity

Definition

A function $f: X \times \Theta \to \mathbb{R}$ is supermodular in x if, for all $x, y \in X$ and $\theta \in \Theta$, we have

$$f(x \vee y, \theta) - f(x, \theta) \geq f(y, \theta) - f(x \wedge y, \theta).$$

Differential Versions

Theorem

If $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is twice continuously differentiable, then f has increasing differences in (x, θ) iff

$\frac{\partial^2 f(x,\theta)}{\partial x_i \partial \theta_j} \geq 0$ for all $x \in X, \theta \in \Theta, i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$,

and f is supermodular in x iff

$$\frac{\partial^2 f(x,\theta)}{\partial x_i \partial x_j} \geq 0 \text{ for all } x \in X, \theta \in \Theta, i \neq j \in \{1, \dots, n\}.$$

Topkis' Theorem

Theorem

If $X \subseteq \mathbb{R}^n$ is a lattice, $\Theta \subseteq \mathbb{R}^m$, and $f: X \times \Theta \to \mathbb{R}$ has increasing differences in (x, θ) and is supermodular in x, then $X^*(\theta)$ is increasing in the strong set order.

There are also multidimensional versions of the Milgrom-Shannon and Edlin-Shannon theorems.

Application 1: Comparative Statics of Input Utilization

Suppose firm has production function $f: \mathbb{R}^n \to \mathbb{R}$, output price p, input price vector q :

$$\max_{y \in \mathbb{R}_+^n} pf(y) - q \cdot y$$

Assume f non-decreasing and supermodular.

f non-decreasing $\Longrightarrow$ objective has increasing differences in $(y, (p, -q))$.

Theorem

Suppose a competitive firm's production function is increasing and supermodular in its inputs. If the price of the firm's output increases and/or the price of any of its inputs decreases, then the firm increases the usage of all of its inputs.

31 (Formally, $Y^*(p, q)$ increases in the strong set order.)

Application 1.5: The Law of Supply

$$\max_{y \in \mathbb{R}_+^n} pf(y) - q \cdot y$$

Can use Topkis' theorem to give alternative proof of law of supply, without any assumptions on f.

Let

$$\begin{aligned} x &= f(y) \\ c(x) &= \min_{y \in \mathbb{R}_+^n : f(y) \geq x} q \cdot y \end{aligned}$$

Rewrite problem as

$$\max_{x \in \mathbb{R}} px - c(x)$$

Problem has increasing differences in (x, p), so $x^*(p)$ increasing in strong set order.

(And every selection from $x^*(p)$ is increasing: see pset.)

Application 2: The LeChatelier Principle

"Firms react more to input price changes in the long-run than the short-run."

Suppose inputs are labor and capital, and capital is fixed in short run.

Seems reasonable that if price of labor changes, firm only adjusts labor slightly in short run, stuck with its old capital usage.

In long run, will adjust labor more, once can choose "right" capital usage.

We give example that shows LeChatelier Principle doesn't always apply, and then use Tokpis to formulate rigorous version of the principle.

Example

Firm can produce $10 of output by using either

1. 2 units of L.
2. 1 unit each of L and K.

Can also shut down and produce nothing.

Initial prices: $2 per unit of L, $3 per unit of K.

Firm produces using 2 units of L.

Suppose price of L rises to $6, K fixed in short run.

In short run, firm shuts down.

In long run, firm produces using 1 unit each of L and K.

In short run, demand for L drops from 2 to 0 .

In long run, goes back up to 1 .

LeChatelier principle fails.

What went wrong?

1 unit of L is complementary with 1 unit of K, but 2 units of L are substitutable with 1 unit of K.

L usage drops from 2 to 0 makes 1 unit of K more valuable ("substitution"), but when K usage rises from 0 to 1 this makes 1 unit of L more valuable ("complementarity").

Suggests LeChatelier principle failed because inputs switched from being complements to substitues at different usage levels.

LeChatelier Revisited

Let

$$
\begin{aligned}
x(y,\theta) &= \arg\max_{x\in X} f(x,y,\theta) \\
y(\theta) &= \arg\max_{y\in Y} f(x(y,\theta),y,\theta)
\end{aligned}
$$

$x(y,\theta)$ is optimal "short-run" x (i.e., holding y fixed).

$y(\theta)$ and $x(y(\theta),\theta)$ are optimal "long-run" choices.

Theorem

Suppose $f: X \times Y \times \Theta \to \mathbb{R}$ is supermodular, $\theta \geq \theta'$, and maximizers below are unique. Then

$$x(y(\theta),\theta) \geq x(y(\theta'),\theta) \geq x(y(\theta'),\theta').$$

Corollary (LeChatelier Principle)

Suppose a firm's problem is

$$\max_{K,L\in\mathbb{R}_+} pf(K,L) - wL - rK$$

with either $f_{KL} \geq 0$ for all (K,L) or $f_{KL} \leq 0$ for all (K,L), and suppose K is fixed in the short-run, while L is flexible.

Then, if the wage w increases, the firm's labor usage decreases, and the decrease is larger in the long-run than in the short-run.