General Equilibrium: short version

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Let's go back to basic consumer theory, without certainty.

Consumer chooses $x \geq 0$ to solve $Max\ u(x)$ subject to $p \cdot x \leq w$.

We get a demand function $x(p)$ for an exogenous p.

Our objective now is to endogenize prices.

Main idea: add a supply = demand condition.

Overview

In the basic model, a consumer has an exogenous monetary income w.

Now we will consider that the consumer has an initial endowment of goods available in this economy.

Consumer $i=1,\dots,I$ has an endowment $e^i = (e_1^i, \dots, e_L^i)$ for each of the L goods in this economy.

This endowment has value $p \cdot e^i$, so he solves now:

$$Max\ u^i(x) \text{ subject to } p \cdot x \leq p \cdot e^i$$

For example, a consumer may have 24 hours per day to allocate between leisure and work. The price of each hour is the wage rate in this economy.

Prices are endogenous $\Longrightarrow$ income is endogenous.

But prices are still exogenous for each consumer: **agents are price takers**.

Prices are the same for everyone:

> "The fact the everyone in the economy faces the same prices is what generates the common information needed to coordinate disparate individual decisions." (Levin)

No production for the moment: **pure exchange economy**.

General (Walrasian) equilibrium takes into account the fact that prices in one market affects decisions in other markets.

> Different from partial equilibrium!
>
> Related to income effect.

Questions:

An equilibrium exists?

What are the properties of an equilibrium?

> (We don't want to describe properties of elements of the empty set.)

Walrasian Equilibrium: Definition

A Walrasian equilibrium is a vector $\left(p, (x^i)_{i=1}^{I}\right)$ such that consumers optimize and markets clear.

Formally:

1. For all $i = 1, \dots, I$, $x^i \in argmax\ u^i(x)$ subject to $x^i \in \mathcal{B}^i(p) = \{x \in \mathbb{R}^+ : p \cdot x \leq p \cdot e^i\}$

2. For all $l = 1, \dots, L$, $\sum_{i=1}^{I} x_l^i \leq \sum_{i=1}^{I} e_l^i$

Prices are always non-negative.

Pareto Optimality

Before we proceed, we need to define a criterion we will use to evaluate allocations.

Definition: Feasibility

An allocation $(x^i)_i$ is feasible if for all l, $\sum_{i=1}^{I} x_l^i \leq \sum_{i=1}^{I} e_l^i$.

Definition: Pareto optimal (or efficient) allocations

An allocation $(x^i)_i$ is Pareto optimal if there is no other feasible allocation $(\hat{x}^i)_i$ such that $u^i(\hat{x}^i) \geq u^i(x^i)$ for all i with strict inequality for at least one consumer.

Traditional disclaimer: Pareto optimality says nothing about distribution.

Assumptions

For every consumer i, we assume:

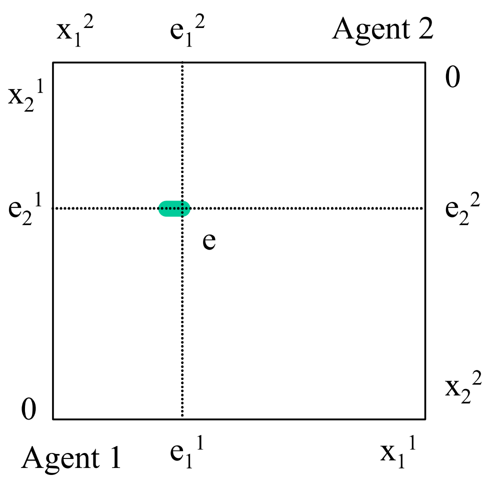
A1: u^i is continuous.

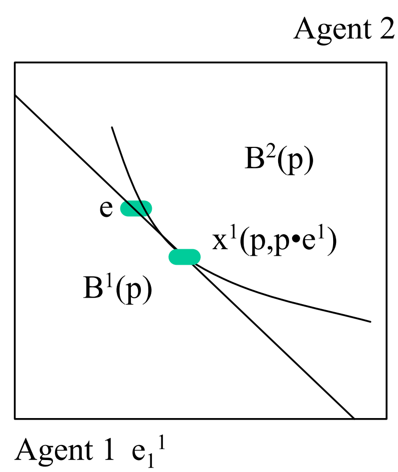
A2: u^i is increasing: $x' \gg x \Longrightarrow u^i(x') > u^i(x)$

A3: u^i is concave

A4: $e^i \gg 0$

Graphical Example: Edgeworth box

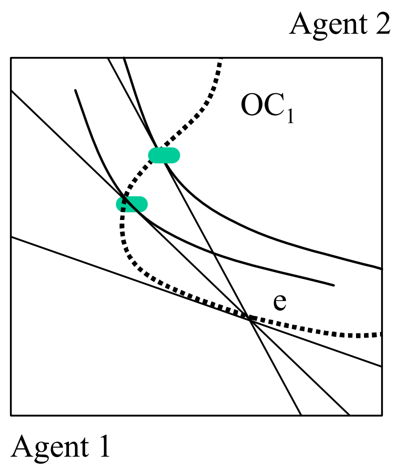




To draw the budget line, we only need relative prices – that is, the slope p_1/p_2.

This is enough because we always have one point that the consumer can afford, for any prices: the initial allocation.

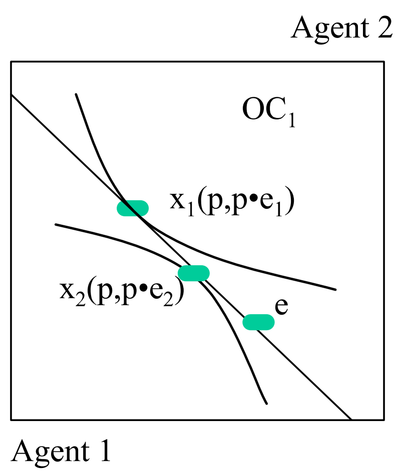
As we change relative prices, we obtain new allocations:



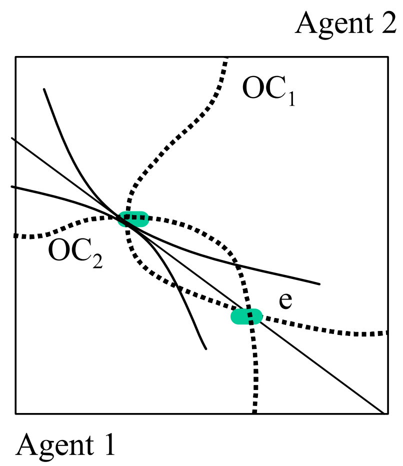
This is the offer curve for an agent.

Walrasian Equilibrium in the Edgeworth Box

Not an equilibrium: excessive demand for one good, excessive supply for the other:



An equilibrium:



Walrasian equilibrium is at intersection of Offer Curves.

It's on the offer curves: it's optimal for both consumers.

It's at the intersection: markets clear.

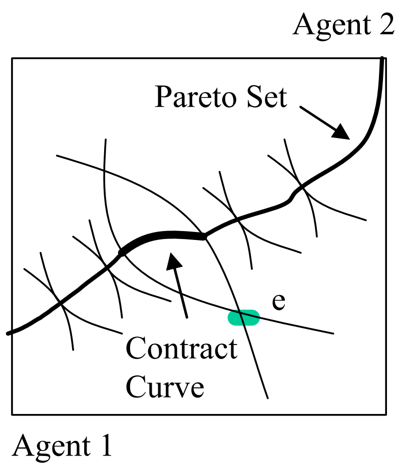
Equilibrium needs not be unique. There may be even no equilibrium.

We have to find conditions for existence and uniqueness.

Pareto optimality in the Edgeworth Box

Pareto Set: set of all Pareto optimal allocations.

Contract curve: subset of the Pareto set that both agents prefer to the initial allocation.



A mutually agreeable bargain that leaves no gains on the table should lead to the Contract Curve.

Indeed, under some conditions, every Walrasian equilibrium is on the Contract Curve. This is the first welfare theorem.

First Welfare Theorem

Every Walrasian equilibrium is Pareto optimal.

Formal statement:

Theorem:

Let $\left(p, \left(x^i\right)_i\right)$ be a Walrasian equilibrium. If agents are price takers, there is a price for each good, and A2 holds, then $\left(x^i\right)_i$ is Pareto optimal.

Proof:

The proof is by contradiction.

Assume $\left(x^i\right)_i$ is a Walrasian equilibrium but is not Pareto optimal.

Then there is an allocation $\left(\hat{x}^i\right)_i$ such that $u^i\left(\hat{x}^i\right) \geq u^i\left(x^i\right)$ for all i and $u^i\left(\hat{x}^i\right) > u^i\left(x^i\right)$ for some i'.

The weak inequality implies $\boldsymbol{p \cdot \hat{x}^i \geq p \cdot x^i = p \cdot e^i}$. Otherwise, we would have $p \cdot \hat{x}^i < p \cdot e^i$: that is, $\hat{x}^i$ is strictly cheaper than x^i, and is weakly preferred to x^i. Hence the consumer could increase slightly the amount of each good to get a new bundle $\hat{x}^i + \varepsilon \cdot (1, \dots, 1)$ that is still cheaper but is strictly preferred to $\hat{x}^i$ (because of A2), and hence is strictly preferred to x^i. Hence x^i cannot be optimal – contradiction, because x^i is a Walrasian equilibrium, hence optimal.

Analogously, **the second inequality implies $\boldsymbol{p \cdot \hat{x}^{i'} > p \cdot x^{i'} = p \cdot e^{i'}}$.** Otherwise, we would have $p \cdot \hat{x}^{i'} \leq p \cdot e^{i'}$. That is, $\hat{x}^{i'}$ would be affordable and strictly preferred to $x^{i'}$ - contradiction because $x^{i'}$ is assumed to be optimal.

So we have:

$$\boldsymbol{p \cdot \hat{x}^i \geq p \cdot x^i = p \cdot e^i}$$

$$\boldsymbol{p \cdot \hat{x}^{i'} > p \cdot x^{i'} = p \cdot e^{i'}}$$

Since prices are non-negative, these conditions imply $\sum_{i=1}^{I} \hat{x}_l{}^i > \sum_{i=1}^{I} x_l^i = \sum_{i=1}^{I} e_l^i$.

Hence $\left(\hat{x}^i\right)_i$ is not feasible – contradiction.

QED

Discussion

The 1st Welfare Theorem states that all mutually beneficial gains may be achieved through trade, given the assumptions:

Basic structure on choices, rationality, monotonicity, complete markets, no market power.

This is Adam Smith's invisible hand: decentralized market is efficient.

It's usual to write it as follows:

Complete markets and absence of market power (plus basic assumptions on preferences) $\Rightarrow$ every Walrasian equilibrium is efficient.

We may write it the other way around:

If a Walrasian equilibrium is not efficient (but basic assumptions hold), then either markets are incomplete, or there is market power.

Incomplete markets: no price for some good, externality, asymmetric information…

Market power: price-making behavior, such as monopolies.

Second Welfare Theorem

Every Pareto efficient allocation may be sustained as a Walrasian equilibrium, if preferences are convex.

Formal Statement:

Theorem:

Assume (A1)-(A4) hold. If $\left(e^{i}\right)_{i}$ is Pareto optimal, then there is a price vector $p \in \mathbb{R}_{+}^{L}$ such that $\left((p),\left(e^{i}\right)_{i}\right)$ is a Walrasian equilibrium.

Proof:

1- We will use (a version of) the separating hyperplane theorem: if $A \subseteq \mathbb{R}^{n}$ is convex and $x \notin A$, then there exists $p \neq 0$ such that $p \cdot a \geq p \cdot x$ for all $a \in A$.

2- Define the following set: $A^i = \{a \in \mathbb{R}^L : e^i + a \geq 0 \ and \ u^i(e^i + a) > u^i(e^i)\}$. This is a set of redistributions a that make agent i strictly better off.

3- Preferences are convex (utility is concave), and hence A^i is convex.

4- Define now $A = \sum_{i=1}^{I} A^i = \{a \in \mathbb{R}^L : a = \sum_{i=1}^{I} a^i \ with \ a^i \in A_i\}$. This is a sum of redistributions.

5- A^i convex implies A convex (exercise).

6- $0 \notin A$. Otherwise there would be $(a^i)_i$ with $\sum_{i=1}^{I} a^i = 0$ and $u^i(e^i + a) > u^i(e^i)$ for all i, meaning that $(e^i)_i$ is not Pareto optimal.

7- The separating hyperplane theorem now implies that there exists $p \neq 0$ such that $p \cdot a \geq p \cdot 0 = 0$ for all $a \in A$. (Point 6 implies that we can take $x = 0$ in point 1.) In short, $p \cdot a \geq 0$

8- $a \gg 0$ implies that $a \in A$ by monotonicity: we can just split the strictly positive amount of each commodity among consumers and make everyone strictly better off due to monotonicity.

9- The two previous points ($p \cdot a \geq 0$ and $a \gg 0$) imply $p \geq 0$. If there were some $p_l < 0$, then we could take a_l very large, a_k very small for all $k \neq l$, and get $p \cdot a < 0$, a contradiction.

10- Points 7 and 9 give us $p \neq 0$ and $p \geq 0$. Hence $p > 0$.

11- Now we need to show that $\left((p), (e^i)_i\right)$ is a Walrasian equilibrium. That is: consumers optimize and markets clear.

12- Market clearing holds by definition since $(e^i)_i$ is the initial allocation.

13- We need to show that $(e^i)_i$ is the optimal demand for prices p. To do that, we will show that if $u^i(x^i) > u^i(e^i)$, then necessarily $p \cdot x^i > p \cdot e^i$: that is, it is not in the budget set, and hence cannot be the solution to the consumer problem.

14- If $u^i(x^i) > u^i(e^i)$, then $p \cdot x^i \geq p \cdot e^i$. If not, then $p \cdot x^i < p \cdot e^i$, and by monotonicity (or simply local non-satiation), it would be possible to find another allocation strictly better than x^i and still affordable.

15- By continuity of the utility function, there is $\lambda < 1$, but very close to one, such that $u^i(\lambda x^i) > u^i(e^i)$ still holds.

16- Repeating the argument in 14 now to λx^i, one has $p \cdot \lambda x^i \geq p \cdot e^i$. But $\lambda < 1$ implies that $p \cdot x^i > p \cdot \lambda x^i$. Putting these two inequalities together, we get $p \cdot x^i > p \cdot e^i$. This is what we needed to show in 13, concluding the proof.

QED.

Characterization of Equilibrium

Let's make differentiability and concavity assumptions to use first order conditions to characterize Pareto optimal allocations.

The idea is to give some intuition for what conditions must be satisfied on the margin at any Pareto optimal allocation, and hence, by the first Welfare theorem, at any Walrasian equilibrium.

We also tie the set of Pareto optimal allocations to the set of allocations that maximize linear Bergson-Samuelson social welfare functions.

One way to identify the set of Pareto optimal allocations $x = (x^1, \dots, x^I)$ is as solutions to the following program:

$$\begin{array}{lll} & \max\limits_{x} & u^1(x_1^1, \dots, x_L^1) \\ \text{s.t.} & u^i(x_1^i, \dots, x_L^i) \geq \bar{u}^i & \text{for } i = 2, \dots, I \\ & \sum\limits_i x_l^i \leq \sum\limits_i e_l^i & \text{for } l = 1, \dots, L. \end{array}$$

The idea here is to maximize the utility of the first consumer subject to feasibility and to the other consumers getting at least some pre-specified level of utility.

By varying the level of required utility for consumers $2, \dots, I$, we can recover the full set of Pareto optimal allocations.

Under assumptions (A1)-(A3), all of the constraints must be binding at the solution

If the utility constraint for i were slack we could reduce x^i by ε in all directions and increase x^1 by the same amount; if the resource constraint were slack we could increase either x^1 or one of the x^i's.

If we assume in addition that each agent has a differentiable utility function, the problem satisfies the conditions of the Kuhn-Tucker theorem

We can use the Kuhn-Tucker conditions to characterize the solution.

Let λ^i denote the Lagrange multiplier on agent i 's constraint and let μ_l denote the constraint on commodity l.

Kuhn-Tucker conditions (eq's 1):

$$\lambda^i \frac{\partial u^i}{\partial x_l^i} - \mu_l \leq 0$$

$$x_l^i \geq 0$$

$$\left(\lambda^i \frac{\partial u^i}{\partial x_l^i} - \mu_l\right) x_l^i = 0$$

For an interior solution, these conditions reduce to:

$$\lambda^i \frac{\partial u^i}{\partial x_l^i} = \mu_l$$

We also have the requirement that each of the $(I-1)+L$ constraints is binding:

$$\begin{aligned} u^i\left(x_1^i, \dots, x_L^i\right) &= \bar{u}^i && \text{for } i = 2, \dots, I \\ \sum_i x_l^i &= \sum_i e_l^i && \text{for } l = 1, \dots, L \end{aligned}$$

Adopt the convention that $\lambda^1 = 1$; you'll see where this bit of notation comes in useful later.

Because each of the constraints binds at the optimum, $\lambda^i > 0$ for $i = 2, \dots, I$ and $\mu_l > 0$ for all l.

The Kuhn-Tucker conditions given in (1) are easy to interpret.

> Recall that λ^i is precisely the marginal value, or shadow price, of consumer i 's income in terms of consumer 1's utility.
>
> That is, at the optimum taking a util away from agent i would allow us to increase agent 1's utility by λ^i.

At the same time, μ_l is the shadow price on commodity l (again in terms of agent 1's utility).

> An extra unit of commodity l would allow us to increase agent 1 's utility by μ_l while holding everyone else's utility constant.

Assuming that each consumer consumes a positive amount of each good at the optimum, so that $x_l^i > 0$ for all i, l, we can derive that at any Pareto efficient allocation, we have the following relationship:

$$MRS_{kl}^i = \frac{\partial u^i / \partial x_k^i}{\partial u^i / \partial x_l^i} = \frac{\mu_k}{\mu_l} = \frac{\partial u^j / \partial x_k^j}{\partial u^j / \partial x_l^j} = MRS_{kl}^j.$$

That is, at the optimum, the marginal rates of substitution of every agent for every commodity pair k, l must be equal to each other and to the ratio of the shadow prices μ_k and μ_l.

> This is precisely the tangency condition from our earlier Edgeworth box picture.

Welfare theorems revisited

Within this simple framework of differentiable concave utility functions, we can link the Pareto optimal allocations to the set of Walrasian equilibria.

Suppose that x is a Pareto optimal allocation as characterized above.

Let $e^i = x^i$ and define prices $p_l = \mu_l$. Given these prices and endowments, consider the optimization problem facing consumer i :

$$\max_{\tilde{x}^i} u^i(\tilde{x}^i)$$
$$\text{s.t. } p \cdot \tilde{x}^i \leq p \cdot e^i$$

Again, we know the budget constraint will bind at the optimum given our assumptions (that's Walras' Law).

Moreover, we can use the Kuhn-Tucker conditions to characterize the optimum.

Letting $\nu^1, \ldots, \nu^I$ denote the Lagrange multipliers on the budget constraints of agents $1, \ldots, I$, the Kuhn-Tucker conditions state that a necessary and sufficient condition for $(x^1, \ldots, x^I; \nu^1, \ldots, \nu^I)$ to solve the I utility maximization problems given prices p is that for all i, l (eq's 2):

$$\frac{\partial u^i}{\partial x_l^i} - \nu^i \cdot p_l \leq 0$$

$$x_l^i \geq 0$$

$$\left(\frac{\partial u^i}{\partial x_l^i} - \nu^i \cdot p_l\right) \cdot x_l^i = 0$$

and in addition, each of the resource constraints bind.

Again, for an interior solution:

$$\frac{\partial u^i}{\partial x_l^i} = \nu^i \cdot p_l$$

If x is a Pareto optimal allocation, one solution is for each agent i to consume $x^i = (x_1^i, \dots, x_L^i)$ with Lagrange multipliers $\nu^i = 1/\lambda^i$.

Because given prices $p_l = \mu_l$ and endowments $e^i = x^i$, there is an exact equivalence between the Kuhn-Tucker conditions of the I utility maximization problems and the Kuhn-Tucker conditions of the earlier Pareto problem.

Therefore it follows that if x is a Pareto optimal allocation, and $\mu_1, \dots, \mu_L$ the commodity shadow prices from the Pareto problem above, then (μ, x) is a Walrasian equilibrium of the economy $\mathcal{E} = \left((u^i)_{i\in\mathcal{I}}, (x^i)_{i\in\mathcal{I}}\right)$.

This is precisely the Second Welfare Theorem.

To obtain the First Welfare Theorem, we go the other way.

If endowments e and prices p are given and each agent maximizes utility, it must be the case at the solution consumption bundles $x^1, \dots, x^I$, (2) holds and each consumer's budget constraint is satisfied.

Then consider the Pareto problem with $\bar{u}^i = u^i(x^i)$ for agents $2, \dots, I$.

It is easy to check that (1) and each of the constraints is satisfied at $x^1, \dots, x^I$ if we define $\mu_l = p_l, \lambda^i = 1/\nu^i$, and $\bar{u}^i = u^i(x^i)$.

Therefore any Walrasian equilibrium is Pareto optimal.

A Social Welfare Function

There is an alternative approach to characterizing Pareto efficient allocations that is sometimes useful.

In this approach, one considers maximizing a linear (Bergson-Samuelson) social welfare function of the form $\sum_i \beta^i u^i$ subject to a resource constraint.

The program is:

$$\max_{x^1,\ldots,x^I} \sum_i \beta^i u^i\left(x_1^i, \ldots, x_L^i\right)$$
$$\text{s.t.} \sum_i x^i \leq \sum_i e^i$$

Given monotonicity of utility functions, the resource constraint will bind at the optimum and the additional Kuhn-Tucker condition for optimality is that for all agents i and commodities l (eq's 3):

$$\beta^i \frac{\partial u^i}{\partial x_l^i} - \delta_l \leq 0$$

$$x_l^i \geq 0$$

$$\left(\beta^i \frac{\partial u^i}{\partial x_l^i} - \delta_l\right) \cdot x_l^i = 0$$

Interior solution:

$$\beta^i \frac{\partial u^i}{\partial x_l^i} = \delta_l$$

Letting $\beta^i = \lambda^i$ and $\delta_l = \mu_l$ we have an exact correspondence between (1) and (3).

Letting $\beta^i = 1/\nu^i$ and $\delta_l = p_l$, we have an exact correspondence between (2) and (3).

So not only do Pareto optimal allocations coincide with Walrasian equilibrium allocations coincide in the sense of the welfare theorems, they coincide with allocations that maximize a linear social welfare function.

Existence of Walrasian Equilibrium

The Model:

There are $i = 1, \dots, I$ agents and $l = 1, \dots, L$ commodities.

Each agent has a utility function $u^i: \mathbb{R}_+^L \to \mathbb{R}$ and an endowment $e^i \in \mathbb{R}_+^L$.

Each agent chooses a consumption bundle $x^i \in \mathbb{R}_+^L$ to maximize u^i subject to $p \cdot x^i \leq p \cdot e^i$, with $p \in \mathbb{R}_+^L$, or analogously we write the restriction as $x^i \in B^i(p)$.

Agents are price takers.

Assumptions:

Preferences are continuous, strictly monotone and weakly convex.

Hence optimal choice needs not be unique, and Marshallian demand does not need to be a function: it may be a correspondence.

That is, a pair (p, w) may be associated to different values $x(p, w)$. The image of (p, w) is a subset of X. In general, we will use the notation $F: X \rightrightarrows Y$ for a correspondence.

For all agents, $e^i \gg 0$.

We define a Walrasian equilibrium as a vector of prices and consumption bundles for each agent $\left(p, \left(x^i\right)_i\right)$ such that:

Choices are optimal: for all i, $x^i \in \underset{x^i \in B^i(p)}{arg\max}\, u^i(p)$

Markets clear: for all l, $\sum_{i=1}^I x_l^i = \sum_{i=1}^I e_l^i$

Roadmap:

1- Define excess demand correspondence:

$$z^l(p) = \sum_{i=1}^{I}\left[x_i^l(p) - e_i^l\right]$$

$$Z(p) = \begin{bmatrix} z^1(p) \\ z^2(p) \\ \vdots \\ z^L(p) \end{bmatrix}$$

2- Recast definition of equilibrium as $Z(p) = 0$ (or more generally $Z(p^*) \leq 0$)

Consumers use Marshallian demand (that is, they optimize) and markets clear.

3- Establish properties of $Z(p)$:

Uhc and non-empty (follows from Theorem of the Maximum)

Convex for all p

Bounded below

If $p_n \to p \neq 0$ with some $p_l = 0$, then $max\{z_1(p_n), \dots, z_L(p_n)\} \to \infty$

Homogenous of degree zero

$pZ(p)$=0 for all p (Walras' Law)

4- Define a (weird) correspondence $m\big(Z(p)\big) = argmax_{\hat{p}\in\Delta^L}\, \hat{p}Z(p)$

Unlike $Z(p)$, domain and codomain are the same

Properties: convex, uhc, non-empty

5- Kakutani's fixed point theorem implies there is p^* such that $m\big(Z(p^*)\big) = p^*$

6- Lastly, show that $Z(p^*) \leq 0$: that is, p^* and associated demands are a Walrasian equilibrium

Equilibrium Existence:

We need the following definition:

$G(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ is the graph of F.

Let's define now a fixed point.

In a simple function $f: \mathbb{R} \longrightarrow \mathbb{R}$, a fixed point is x such that $x = f(x)$. For example, if $f(x) = x^2$, then $x = x^2 = 1$ is a fixed point.

We may generalize it for a correspondence. $F: X \rightrightarrows Y$. Now we write $x \in F(x)$ for a fixed point.

We need to find conditions that guarantee that a fixed point exists.

Kakutani's Fixed Point Theorem

Let $X \subset \mathbb{R}^n$ compact, convex, non-empty. Let $F: X \rightrightarrows X$ such that $G(F)$ is closed, and $F(x)$ is convex for all $x \in X$. Then this correspondence has a fixed point $x^* \in F(x^*)$.

There is another way to present the same result.

Consider again $X \subset \mathbb{R}^n$ compact, convex, non-empty. Let $F: X \rightrightarrows X$ be non-empty, convex, and upper-hemicontinuous (uhc). Then this correspondence has a fixed point $x^* \in F(x^*)$.

Definition of uhc. Consider X, $Y \subset \mathbb{R}^n$ compact and convex. Let $F: X \rightrightarrows Y$. Consider $\{x_n\} \subset X$, $\{y_n\} \subset Y$ and $x_n \longrightarrow x$, $y_n \longrightarrow y$ and $y_n \in F(x_n)$ for all n. Then $y \in F(x)$.

Uhc is equivalent to closed graph only if Y is compact, which is our case.

We will also need the following result:

Maximum Theorem:

Assume $C: Q \rightrightarrows \mathbb{R}^N$ is a continuous correspondence, and $c(q)$ is compact and non-empty for all $q \in Q$. Assume $f: \mathbb{R}^N \to \mathbb{R}$ is a continuous functions. Consider the problem $\max_{x \in c(q)} f(x)$. Then the maximizer $x^*(q)$ is upper-hemi continuous and the value function $f\big(x^*(q)\big)$ is continuous.

Let's go back to the consumer problem (CP). Marshallian demand: $x^i(p) = x(p, pe)$.

Notice that $x^i(\lambda p) = x(\lambda p, \lambda pe) = x(p, pe) = x^i(p)$.

> The first and the last equalities use the definition of $x^i(p)$ in the previous paragraph.
>
> The second equality uses the fact that Marshallian demand is unchanged if we multiply both prices and income by a positive number.
>
> This means that $x^i(p)$ is homogenous of degree zero in prices.

Moreover, strictly monotone preferences implies that the consumer spends all his income: $px^i(p, pe) = pe$, and hence $p\big[x^i(p, pe) - e\big] = 0$.

This holds for any vector of prices p.

Define <u>excess demand</u>:

$$z^l(p) = \sum_{i=1}^{I} \big[x_i^l(p) - e_i^l\big].$$

Define the following notation:

$$Z(p) = \begin{bmatrix} z^1(p) \\ z^2(p) \\ \vdots \\ z^L(p) \end{bmatrix}$$

This is a vector of excess demand, for each good, at prices p.

Another way to define a Walrasian equilibrium:

It is a vector $\left(p, \{x_l^i\}_{l,i}\right)$ such that $Z(p) = 0$.

This summarizes two conditions: consumers optimize (implicit in the use of Marshallian demands) and markets clear.

One may also use $Z(p) \leq 0$, with $Z(p) < 0$ for prices equal to zero.

Notice that if $Z(p) = 0$, then $Z(\lambda p) = 0$ for all $\lambda > 0$.

That is, excess demand is also homogenous of degree zero in prices: this property is inherited from $x^i(\lambda p) = x^i(\lambda p)$, seen above, since endowment e_i^l is constant (i.e., does not depend on prices).

This also implies $p \cdot Z(p) = 0$ for any vector of prices: this is Walras' Law.

This has an important interpretation: we only find relative prices in equilibrium.

Turn now to our equilibrium result.

The function $Z(p)$ has the following properties:

Uhc and non-empty (follows from Theorem of the Maximum). There is only one detail: we cannot apply the theorem directly to CP because $B^i(p)$ is not compact for prices going to zero.

To solve this, simply define that the budget set is $B^i(p) \cap T$, in which:

$T = \left\{x \in \mathbb{R}_+^L : x \leq 2 \cdot \sum_{i=1}^I e^i\right\}$.

CP is unchanged but now we have a compact set, and can apply the Maximum Theorem to conclude that the solution $x^i(p)$ is Uhc and non-empty.)

Convex for all p (direct consequence of result from consumer theory: solution is convex if preferences are convex)

Bounded below: there is $Z > 0$ such that for all l and all p, $z_l(p) > -Z$. (Lowest possible demand is $x_l = 0$, and hence $z_l \geq -\sum_{i=1}^{I} e_i^l$ for all goods.)

If $p_n \to p \neq 0$ with some coordinate $p_l = 0$, then $max\{z_1(p_n), \dots, z_L(p_n)\} \to \infty$. (If some but not all prices go to zero, some consumer must have wealth going to infinity, and then strongly monotone preferences imply that his demand for the good $p_l = 0$ goes to infinity.)

Homogenous of degree zero (shown above)

$pZ(p)$=0 for all p (Walras' Law – shown above)

And these properties imply that there is a Walrasian equilibrium as defined above.

Let us show this.

Begin normalizing prices. We can always do this in general equilibrium because, as we saw, we only look at relative prices. Choose the following normalization:

$$p_1 + p_2 + \cdots + p_L = 1$$

That is, divide all prices by the sum of prices, which must be non-negative: $p_l \geq 0$ for all l.

This is a simplex. The set of all prices that respect these restrictions is Δ^L. That is, $p \in \Delta^L$. This set is closed, bounded (hence compact), and convex.

(Consider the case of two goods for visualization.)

This simplex will be our domain and codomain X.

Now let us define the following function:

$$Z: \Delta^L \rightrightarrows \mathbb{R}^L$$

For each price vector p, $Z(p)$ informs the excess demand for each good l.

This is a correspondence that takes from Δ^L to $\mathbb{R}^L$. Not enough to use Kakutani's theorem because domain and codomain are not the same.

Define the following function:

$$\boldsymbol{m(Z(p)) = argmax_{\hat{p}\in\Delta^L}\, \hat{p}Z(p)}$$

Notice that $m(Z)$ is a vector of prices: it is the vector of prices that maximizes $\hat{p}Z(p)$.

Observe that we first have $\hat{p}$ and then have p: we are choosing $\hat{p}$ to maximize the value of excess demand at the original vector price p.

This is a continuous function on a compact domain, and hence the solution set is non-empty.

So we use the correspondence Z to go from Δ^L to $\mathbb{R}^L$, and then apply $m(Z)$ to go from $\mathbb{R}^L$ to Δ^L. This is a correspondence composition: $\Delta^L \rightrightarrows \mathbb{R}^L \rightrightarrows \Delta^L$.

Let us present now a series of results.

Lemma 1. $m(Z)$ is convex.

Proof. Consider $p_1, p_2 \in \Delta^L$ such that $p_1, p_2 \in argmax_{\hat{p}\in\Delta^L}\, \hat{p}Z(p)$.

Hence $p_1 Z(p) = p_2 Z(p)$. Then for all $\lambda \in [0,1]$:

$$\lambda p_1 Z(p) + (1-\lambda) p_2 Z(p) = p_1 Z(p) = p_2 Z(p)$$

That is, $\lambda p_1 + (1-\lambda)p_2$ also maximizes $\boldsymbol{\hat{p}Z(p)}$.

Hence $\lambda p_1 + (1-\lambda)p_2 \in m(Z)$.

We conclude that $m(Z)$ is convex, since it contains any convex combination of two of its elements. QED.

Lemma 2. $m(Z)$ is uhc.

Proof. Take some sequence $p^n \to p^*$. Consider $Z^n \to Z^* = Z(p^*)$ and $p^n \in m(Z^n)$. We need to show that $p^* \in m(Z^*)$.

Assume $p^* \notin m(Z^*)$.

Then there is some $\bar{p} \neq p^*$ such that $\bar{p}Z^* > p^*Z^*$. This is simply the definition of $m(Z^*)$.

But $Z^n \to Z^*$ and $p^n \to p^*$. Hence $\bar{p}Z^n \to \bar{p}Z^*$. Also $p^n Z^n \to p^* Z^*$.

So we may choose n big enough such that:

$$|\bar{p}Z^n - \bar{p}Z^*| < \frac{\varepsilon}{2}$$

$$|p^nZ^n - p^*Z^*| < \frac{\varepsilon}{2}$$

Remember now that $\bar{p}Z^* > p^*Z^*$. This allows us to conclude:

$$\bar{p}Z^n > p^*Z^* + \frac{\varepsilon}{2}$$

$$p^*Z^* + \frac{\varepsilon}{2} > p^nZ^n$$

The first line is implied by two facts: $\bar{p}Z^n$ is close to $\bar{p}Z^*$ (inequality in yellow), and $\bar{p}Z^* > p^*Z^*$.

The second line is a direct implication of the inequality in green.

These two last lines imply:

$$\bar{p}Z^n > p^nZ^n$$

That is, p^n does not maximize p^nZ^n, which is absurd: we assumed that p^n maximizes p^nZ^n. QED.

This is argument is similar to: $a_n \geq 0$ and $a_n \to a$ imply $a \geq 0$. This argument is simple: the difficulty is basically notation and an unusual environment, not mathematics.

Now let us build the following correspondence $g\colon \Delta^L \rightrightarrows \Delta^L$ as follows:

$$g(p) = m(Z(p))$$

That is, we have $\Delta^L \rightrightarrows \mathbb{R}^L \rightrightarrows \Delta^L$. We may apply Kakutani on the correspondence g.

We have the following result: the composition of non-empty, uhc and convex correspondences is also non-empty, uhc and convex. (Exercise.)

Hence we may apply Kakutani's fixed point theorem to conclude that there is $p^* \in g(p^*)$.

Lastly, we need to show that p^* is a vector of equilibrium prices.

Write the definition of $p^* \in g(p^*)$:

$$p^* \in argmax_{\hat{p} \in \Delta^L} \hat{p}Z(p^*)$$

This means:

$$p^* Z(p^*) \geq pZ(p^*)$$

For all $p \in \Delta^L$. Keep this in mind: we will use it in the last line of the proof below.

Let us prove our last lemma.

Lemma 3. $Z(p^*) \leq 0$.

Proof. Suppose that there is a commodity l such that $Z_l(p^*) > 0$.

Choose $\tilde{p} = (0, \dots 0,1,0, \dots ,0)$: that is, $\tilde{p}_l = 1$, and $\tilde{p}_k = 0$ for $k \neq l$.

Then:

$$\tilde{p}Z(p^*) = Z_l(p^*) > 0 = p^*Z(p^*)$$

The last equality is Walras' Law, which applies for any vector of prices – in particular, it applies for p^*.

This is absurd because $p^*Z(p^*) \geq pZ(p^*)$ for all $p \in \Delta^L$. QED.

Production in General Equilibrium

Everything we have done so far has been for the special case of an exchange economy where goods simply come from nowhere as endowments.

Easy to incorporate firms and production into our general equilibrium model, so long as we assume:

(1) no increasing returns to scale

(2) perfectly competitive price-taking firms.

In this section, we outline the more general Arrow-Debreu model with production, revisit the welfare theorems and equilibrium existence, and then consider some simple examples.

Adding Production to the Model

Consumers $i = 1, \ldots, I$ of the earlier model, with utility functions $u^1, \ldots, u^I$.

K firms $k \in \mathcal{K}$ with production sets $Y^k \in \mathbb{R}^N$.

Each Y^k is a set of production plans: if $y \in Y^k$, then $y_l < 0$ means good l is being used as an input; $y_l > 0$ means good l is being produced as an output.

Firms are owned by the households.

Let α^{ki} denote i 's share of firm k.

A production economy is then:

$$\mathcal{E}=\left(\left(u^i,e^i,\left(\alpha^{ki}\right)_{k\in\mathcal{K}}\right)_{i\in\mathcal{I}},(Y^k)_{k\in\mathcal{K}}\right).$$

Firm k takes prices $p\in\mathbb{R}^N$ as given and choose a production plan $y^k\in Y^k$ to solve:

$$\max_{y\in Y^k} p\cdot y.$$

Definition 6 A Walrasian equilibrium is a vector $\left(p,\left(x^i\right)_{i\in\mathcal{I}},(y^k)_{k\in\mathcal{K}}\right)$ such that

Firms maximize profits: for all $k\in\mathcal{K}$,

$$y^k\in\arg\max_{y\in Y^k} p\cdot y$$

Consumers maximize utility: for all $i\in\mathcal{I}$,

$$\begin{aligned} x^i\in\ &\arg\ \max_x u^i(x)\\ &\text{s.t. } p\cdot\left(x-e^i\right)-p\cdot\sum_{k\in\mathcal{K}}\alpha^{ki}y^k\le 0\end{aligned}$$

Markets clear:

$$\sum_{i\in\mathcal{I}}\left(x^i-e^i\right)-\sum_{k\in\mathcal{K}}y^k=0.$$

Assumptions about Production

We'll want to make some assumptions on Y^k to ensure that an equilibrium exists with production.

The simplest such assumption is that Y^k is convex and compact for all firms k, but it seems unreasonable to assume that a production set is bounded.

Instead, we assume:

(A5) For all firms $k \in \mathcal{K}, Y^k$ is closed and convex.

(A6) For all firms $k \in \mathcal{K}, 0 \in Y^k$ and $\mathbb{R}^N_{--} \subset Y^k$.

These assumptions rule out increasing returns to scale.

> If $y \in Y^k$, then so is βY^k for any $0 < \beta < 1$.
>
> So it is always possible to "scale" down production or break it up into arbitrarily small productive units.

We need one further assumption to ensure that firms cannot cooperate in a clever way and produce an infinite amount of goods - i.e. to ensure that one firm doesn't produce 1 pound of iron into 1 pound of steel, while another firm produces 2 pounds of iron from that 1 pound of steel.

Debreu (1959) makes an assumption directly on the aggregate production possibilities:

(A7) If $Y = \sum_{k \in \mathcal{K}} Y^k$, then $Y \cap -Y = \{0\}$.

Think about why this rules out the above story. With these assumptions in place, our earlier welfare and existence results carry through.

Efficiency and Existence

The definition of feasibility and Pareto efficiency carry through immediately to the case of production.

Definition 7 An allocation and production plan $\left((x^i)_{i \in \mathcal{I}}, (y^k)_{k \in \mathcal{K}}\right)$ is feasible if $\sum_{i \in \mathcal{I}} \left(x^i - e^i\right) - \sum_{k \in \mathcal{K}} y^k \leq 0$.

Definition 8 A feasible allocation and production plan $\left((x^i)_{i \in \mathcal{I}}, (y^k)_{k \in \mathcal{K}}\right)$ is Pareto efficient if there is no other feasible allocation and production plan

$\left((\hat{x}^i)_{i\in\mathcal{I}}, (\hat{y}^k)_{k\in\mathcal{K}}\right)$ satisfying $u^i(\hat{x}^i) \geq u^i(x^i)$ for all i, with strict inequality for at least one i'.

We now state the two welfare theorems.

Theorem 12 (First Welfare Theorem) Assume $\mathcal{E}$ is a production economy that satisfies (A2). If $\left(p, (x^i)_{i\in\mathcal{I}}, (y^k)_{k\in\mathcal{K}}\right)$ is a Walrasian equilibrium for $\mathcal{E}$, then $\left((x^i)_{i\in\mathcal{I}}, (y^k)_{k\in\mathcal{K}}\right)$ is Pareto efficient.

The proof is virtually identical to the exchange case.

Theorem 13 (Second Welfare Theorem) Assume utility functions and production sets satisfy (A2)-(A5) and that $\left((x^i)_{i\in\mathcal{I}}, (y^k)_{k\in\mathcal{K}}\right)$ is a Pareto efficient allocation. Suppose $x^i \gg 0$ for all i. Then there is a price vector $p > 0$, ownership shares $(\alpha^{ki})_{i,k}$, and endowments $(e^i)_i$ such that $\left(p, (x^i)_{i\in\mathcal{I}}, (y^k)_{k\in\mathcal{K}}\right)$ is a Walrasian equilibrium given these endowments and ownership shares.

The proof again relies on the Separating Hyperplane Theorem; you can check it out in MWG.

Key assumption is convexity of the production possibility sets. This is what enables us to find a separating hyperplane between the set of feasible production plans and the aggregate "better than" set. One then shows that the separating hyperplane is a supporting price vector.

What about equilibrium existence? If we impose all three of the Assumptions above, we're in good shape.

Theorem 14 (Existence of Equilibrium) Assume $\mathcal{E}$ is a production economy satisfying (A1)-(A7). Then there exists a Walrasian equilibrium of $\mathcal{E}$.

Linear Activity Analysis

If we are modeling production, we not only have to pick utility functions but also production sets or production functions. A simple case is the so called 'linear activity model' of production. In this model, all production sets are convex cones spanned by finitely many rays. In particular, there is only one firm (this actually won't make any difference - see below). The firm has access to M linear activities $a_m \in \mathcal{M} \subset \mathbb{R}^L$. It can operate each activity at some level $\gamma \geq 0$. The production set Y is the convex hull of these activities,

$$Y = \left\{ y \in \mathbb{R}^L : y = \sum_{m=1}^{M} \gamma_m a_m \text{ for some } \gamma \in \mathbb{R}_+^M \right\}.$$

Our assumption of free disposal is satisfied if the vectors

$$(-1,0, \dots 0), (0, -1,0, \dots 0), \dots (0, \dots ,0, -1)$$

are all in $\mathcal{M}$. Figure 11 shows the special case of 4 activities and 2 goods. There are two productive activities: activity 1 allows 2 units of good 2 to be converted into 1 unit of good 1. Activity 2 allows 3 units of good 1 to be converted into 1 unit of good 2. Also there are two "free disposal activities". Therefore:

$$\mathcal{M} = \{(1, -2), (-3,1), (0, -1), (-1,0)\}$$

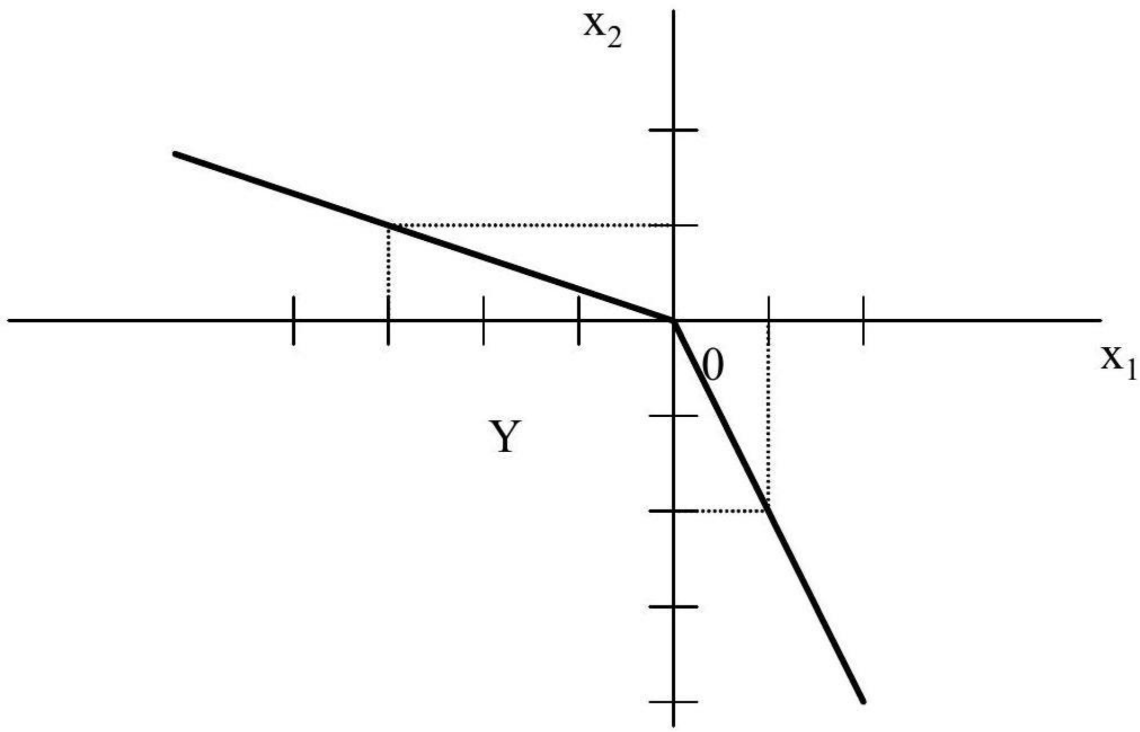


Figure 11: Activity Analysis Model

In the activity analysis model, given a price vector p, a profit maximizing production plan exists if and only if $p \cdot a_m \leq 0$ for all $m = 1, \dots, M$. If $p \cdot a_m > 0$ for some $m = 1, \dots, M$, the firm could choose $\gamma_m \rightarrow \infty$ and make infinite profits. Also, if $p \cdot a_m < 0$ for some m it is clear that the optimal $\gamma_m = 0$.

This simple observation already tells us a lot about what kind of prices could potentially be equilibrium prices. Indeed, in many cases the equilibrium prices will just be determined by the zero-profit conditions, with utility maximization and market clearing pinning down the levels at which the activities are operated.

An important thing to note is that if all production sets are of this simple linear form, firms do not play a role at all. As there will never be any equilibrium profits, what matters is just the aggregate production set. Whether we interpret each activity as a separate firm or we assume that one firm owns all the activities, or even that several firms operate different sets of overlapping activities makes no difference as long as we stay in our competitive paradigm. [3] This constant returns property is shared by the Cobb-Douglas production model that you have probably seen a lot in macroeconomics.

An Example with Numbers

To make things really concrete, let's consider an example with numbers. Suppose there are two agents and three goods. The agents have identical utility functions:

$$u^i(x) = \log(x_1) + \log(x_2) + \log(x_3)$$

Endowments are $e^1 = (1,2,3)$ and $e^2 = (2,2,2)$. Suppose that there are two activities $a_1 = (2,-1,0.5)$ and $a_2 = (0,1,-1)$.

What does the Walrasian equilibrium look like? Let's normalize $p_3 = 1$. Now, if activity 2 is used in equilibrium, it must be the case that (by zero profit) $p_2 = 1$. Similarly, if activity 1 is used in equilibrium, then $p_1 = 0.25$. These prices are upper bounds on the equilibrium prices if these activities are not used in equilibrium.

Let's see if we can find an equilibrium where both activities are used. Given prices $p = (0.25,1,1)$, we solve the utiliy maximization problem for agent i. This gives us:

$$\frac{1}{p_1 x_1^i} = \frac{1}{p_2 x_2^i} = \frac{1}{p_3 x_3^i} \text{ and } \sum_l p_l x_l^i = \sum_l p_l e_l^i$$

Plugging in our price vector and the endowments, we have:

$$\frac{4}{x_1^i} = \frac{1}{x_2^i} = \frac{1}{x_3^i} \text{ and } \frac{1}{4}x_1^i + x_2^i + x_3^i = w^i$$

where $w^1 = 5.25$ and $w^2 = 4.5$. Therefore:

$$x^1 = (7,1.75,1.75)\; x^2 = (6,1.5,1.5)$$

Therefore aggregate demand is $(13,3.25,3.25)$. The aggregate endowments are $(3,4,5)$, so the only way we can have market clearing is if the aggregate production is $(10, -0.75, -1.75)$. This isn't a problem. The firm will simply operate activity 1 at a level $\gamma_1 = 5$ and operate activity 2 at a level $\gamma_2 = 4.25$.

General Equilibrium with Uncertainty

Our goal in this last section is to introduce time and uncertainty into the basic model.

Introducing uncertainty allows a role for financial markets.

We first discuss the basic framework, then look at a model with financial markets and a single consumption good. In the context of this simple model, we consider what it means for there to be an absence of arbitrage possibilities. We also look at why the first welfare theorem can fail if there are too few financial securities.

Modeling Uncertainty and Time

Among the many simplifications of the Arrow-Debreu model we have studied so far is that it's essentially a static model with no uncertainty at all.

> Ideally, we'd like to include both time and uncertainty into our model of competitive trade.

Introducing time into the model isn't too hard. A tomato in summer is a different good than a tomato in winter. So perhaps we can just think about a commodity as being identified not only by its physical characteristics but also by its date.

Uncertainty seems more complicated, but a brilliant modelling innovation of Arrow (1953) comes to the rescue.

Arrow's insight was to introduce "states of the world" along the lines of Savage's decision theory.

> A state of the world is a complete description of a date-event.

Unlike in Savage, however, we're going to assume that these states aren't personal and subjective

Instead everyone somehow agrees on the possible states (there could be a lot).

People don't have to agree on the probabilities of the states occurring, though that is often assumed.

We now think about the general model as having a finite number of time periods.

In each period there is a set of possible states and there can be uncertainty about what state will arise at date $t + 1$ - the probabilities can even depend on what state was realized at date t.

With these ideas in mind, we can think about re-interpreting our Walrasian model as follows.

We model uncertainty as an event tree with S nodes, $\xi \in \Xi$. We denote a node's predessor by ξ_- and its set of successors by $\Upsilon(\xi)$.

At each t we summarize the nodes in this period in a set $\mathcal{N}_t$.

We denote the root node by 0.

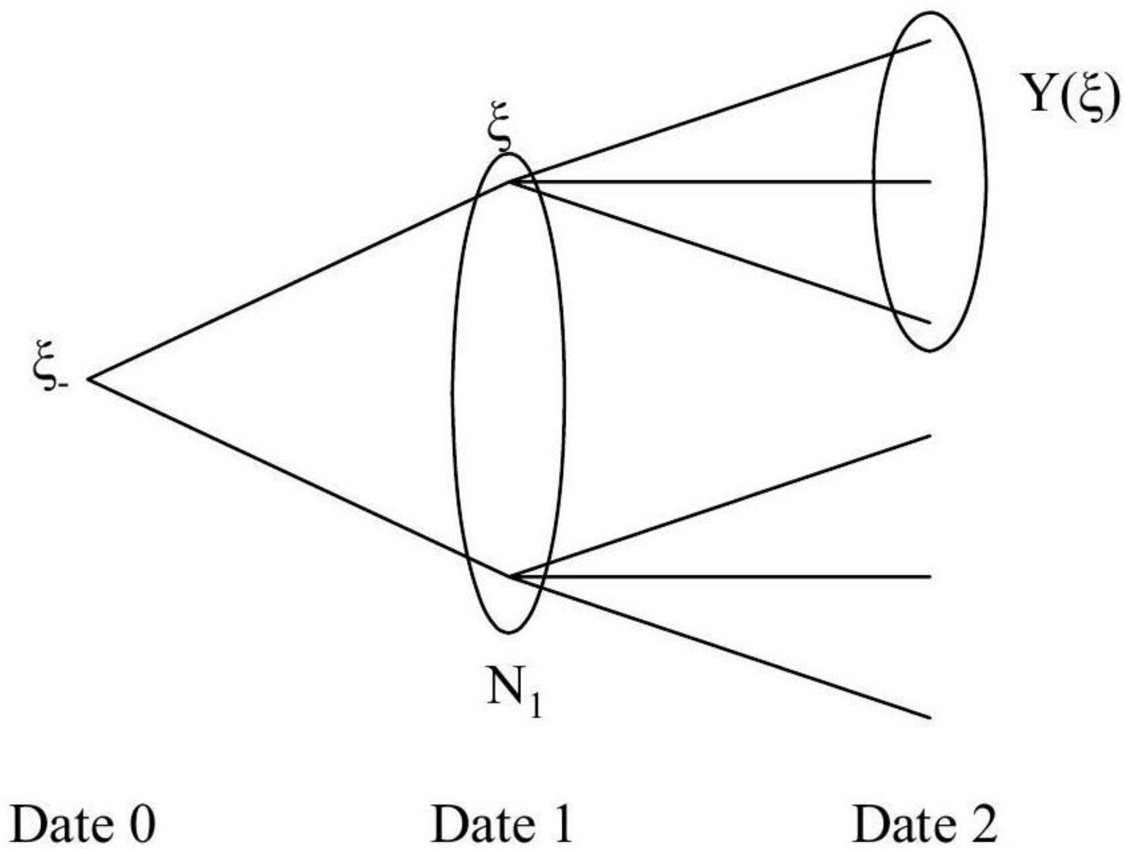


Figure 12: An Event Tree

There are L commodities at every node so the total number of commodities is SL.

There are I agents. Each has an endowment $e^i \in \mathbb{R}^{SL}_{++}$.

Agent i 's consumption set is $\mathbb{R}^{SL}_{+}$ and his utility function is $u^i: \mathbb{R}^{SL}_{+} \to \mathbb{R}$.

The utility function may or may not satisfy the von-Neumann Morgenstern axioms.

Define a Walrasian equilibrium exactly as before: a set of prices and resulting allocation such that (i) all agents maximize utility given prices; and (ii) markets clear.

The idea here is that all trades take place at date zero and there is no retrading in later periods.

Under our earlier assumptions on preferences, a Walrasian equilibrium exists, and the Welfare Theorems hold.

This is the model in chapter 7 of Debreu's Theory of Value. As Debreu puts it: "A contract for the transfer of commodities now specifies, in addition to its physical properties, its location and date, an event on the occurrence of which the transfer is conditional. **This new definition of commodity allows one to obtain a theory of uncertainty free from any probability concepts and formally identical to the theory of certainty.**"

This is quite elegant, but as Arrow originally pointed out, it seems unrealistic that all these contingent trades would occur at date 0.

Instead, what tends to happen is that there are financial securities that are traded on exchanges and some of these securities pay out contingent on certain events (e.g. hurricane insurance pays out contingent on there being a hurricane; stocks pay dividends contingent on company performance).

Arrow cleverly reformulated the model as follows.

Assume at each node ξ there are spot markets for the L commodities at that node.

Assume that these commodities have prices $p(\xi)$.

At node 0 there are now $(S-1)$ Arrow securities (i.e. one for each future node), where an Arrow security ξ pays one unit of good one at node ξ.

An equilibrium is now defined as utility maximization and market clearing at each node and in the $S-1$ markets for Arrow securities at date 0.

The amazing result is that even though there are only $S-1$ securities in the economy, the Arrow-Debreu allocation obtains.

The result isn't even that hard to prove, though we won't do it now.

The next question that arose (in a paper by Radner, 1972) was the following: what happens if there are securities that pay out in future contingencies (like stock in different companies), but not a complete set of Arrow-Debreu securities.

This makes for arguably a more realistic model of actual securities markets.

This question has given rise to a large "incomplete markets" literature in economics and finance.

One of the interesting results from this literature is that without a complete set of A-D securities, the first welfare theorem generally won't hold.

So there is potentially room for government intervention and policy questions become interesting.

A Simple Finance Model

In this section we introduce and study what is just about the simplest general equilibrium model with uncertainty. We assume there are two periods and in each state of the world there is just one consumption good. We normalize the spot price in each period to be equal to one.

We assume there are $S+1$ states of the economy. At time $t=0$ the economy is in state $s=0$; at time $t=1$ the economy can be in one of S possible states. In each state $s=0,\dots,S$ there is a single perishable consumption good.

Each agent $i \in \mathcal{I}$ has an initial endowment $e^i = \left(e_0^i, \dots, e_S^i\right) \in \mathbb{R}_{++}^{S+1}$ and has a utility function $u^i \colon \mathbb{R}_{+}^{S+1} \to \mathbb{R}$ over consumption bundles $c^i = \left(c_0^i, \dots, c_S^i\right) \in \mathbb{R}_{+}^{S+1}$. We asume each agent's utility function satisfies the standard assumptions - it's increasing, continuous and strictly concave. Also, let's define $\bar{x} = (x_1, \dots, x_S)$ as the $t=1$ part of the vector $x = (x_0, x_1, \dots, x_S)$. The aggregate endowment is $e = \sum_{i \in \mathcal{I}} e^i$.

There are J assets or securities. Each asset j pays dividends at date $t=1$ which we denote by $d^j \in \mathbb{R}^S$. The price of asset j at time $t=0$ is q_j. Without loss of generality we assume that these assets are in zero net supply (if we wanted the assets to be stock in some firm, there would be positive net supply, but then we could put the dividends into agent's endowments and be back to zero net supply). We collect all assets' dividends in the matrix:

$$A = (d^1, \dots, d^J) \in \mathbb{R}^{S \times J}$$

At time $t = 0$, each agent i chooses a portfolio $\alpha^i \in \mathbb{R}^J$, where α_j^i is the amount of asset j held by agent i. An agent's portfolio uniquely defines his wealth at each time one state, and hence his consumption (recall that prices are normalized to one at each date one state): $\bar{x}^i = e^i + A\alpha^i$ and $x_0^i = e_0^i - \alpha^i \cdot q$. The net demand of each agent $\bar{x}^i - e^i$ belongs to the span of the asset payoff matrix A :

$$\langle A \rangle = \{z \in \mathbb{R}^S : \exists \alpha \in \mathbb{R}^J \text{ s.t. } z = A\alpha\}$$

A finance economy is hence a triple: $\mathcal{E} = \left((u^i, e^i)_{i \in \mathcal{I}}, A \right)$. Without loss of generality, we can assume that rank $(A) = J$ so there are no redundant assets. With redundant assets, an arbitrage argument would imply that the price of some assets would be uniquely determined by the price of other assets, regardless of preferences. We say that markets are incomplete if $J < S$.

Asset prices are said to be arbitrage-free if it is not possible to achieve a positive income stream in all states by trading at the going prices, i.e. if there is no position $\alpha \in \mathbb{R}^J$ with $q\alpha \leq 0$ and $A\alpha \geq 0$ with one inequality being strict. Here $q\alpha$ is the cost of portfolio α at date 0 and $A\alpha$ is the vector of payoffs at different date one staes. No arbitrage means you can't guarantee positive future income tomorrow without making a positive investment today.

If agents have strictly increasing utility functions, asset prices must preclude arbitrage or there would be a real problem with utility maximization. The absence of arbitrage is thus often seen the fundamental concept in finance (more so than equilibrium). Many important concepts (such as Black-Scholes option pricing) rely solely on arbitrage arguments.

Theorem 15 An asset price vector $q \in \mathbb{R}^J$ precludes arbitrage if and only if there exists a state price vector $\pi \in \mathbb{R}^S_{++}$ such that $q = \pi' \cdot A$.

Proof. Let $M = \{(-q\alpha, A\alpha): \alpha \in \mathbb{R}^J\}$ be the marketed subspace of $\mathbb{R}^{S+1}$. That is, $(-x_0, \bar{x}) \in M$ means that by spending x_0 at date 0 , an agent can ensure the vector of returns $\bar{x}$ at date one. There is no arbitrage if and only if $\mathbb{R}^{S+1}_+ \cap M = \{0\}$. If $(x_0, \bar{x}) \in M$ and $x_0 \geq 0, \bar{x} \geq 0$ with a strict inequality (so $(x_0, \bar{x}) \in \mathbb{R}^{S+1}_+ - \{0\}$, it would be possible to start with zero wealth, consume x_0 today and consume $\bar{x}$ tomorrow - i.e. arbitrage would be possible.

For one direction of the proof, suppose there exists a strictly positive state price vector $\pi \in \mathbb{R}^S_{++}$ such that $q = \pi' A$. We show that this means there is no arbitrage. If there were also a vector $x \in \mathbb{R}^{S+1}_+ \cap M$ with $x \neq 0$, then because $x \in \mathbb{R}^{S+1}_+$ and $\pi \in \mathbb{R}^S_{++}$, we have $(1, \pi) \cdot x > 0$. But by the fact that $x \in M$ and $q = \pi' A$, we also have $(1, \pi) \cdot x = -q\alpha + q\alpha = 0$, a contradiction. Hence, a strictly positive state price vector implies no arbitrage.

For the converse direction, suppose no arbitrage: $\mathbb{R}^{S+1}_+ \cap M = \{0\}$. We use the separating hyperplane to derive a supporting state price vector. Note that M and $\mathbb{R}^{S+1}_+$ are both convex sets whose intersection includes only the point $\{0\}$. The SHT asserts the existence of a vector $\mu \neq 0$ such that $\mu \cdot x < \mu \cdot z$ for all $x \in M$ and all non-zero $z \in \mathbb{R}^{S+1}_+$. [4]

Now, by the definition of M, it must be the case that if $x \in M$ then $-x \in M$,so we must have $\mu \cdot x = 0$ for all $x \in M$. Therefore $\mu \cdot z > 0$. The latter implies that $\mu \gg 0$ (if $\mu_l \leq 0$ for some l, we could find $z \in \mathbb{R}^{S+1}_+ - \{0\}$ with $z_l > 0$ and $z_k = 0$ leading to the contradiction $\mu \cdot z \leq 0$). Therefore $-\mu_1 q + (\mu_2, \dots, \mu_{S+1})A = 0$ and $\pi_s = \mu_{s+1}/\mu_1$ will give us a state price vector (note that to form π we just normalize the prices $-\mu$ has the right relative prices).

Q.E.D.

A lot of asset pricing theory has to do with finding the right state-price vector π. Its existence is ensured by the absence of arbitrage, but often little can be said about it in general models. [5]

Definition 9 A financial markets equilibrium for a finance economy $\mathcal{E}$ is a collection of portfolios $\alpha^* = (\alpha^{1*}, \ldots, \alpha^{I*}) \in \mathbb{R}^{IJ}$, individual consumptions $\left(x^i\right)_{i\in\mathcal{I}}$ and prices $q^* \in \mathbb{R}^J$ such that:

Agents maximize utility:

$$\left(x^i, \alpha^{i*}\right) \in \arg \max_{\alpha^i \in \mathbb{R}^J, c^i \in \mathbb{R}_+^{S+1}} u^i\left(c^i\right)$$
$$\text{s.t. } c^i = e^i + \begin{pmatrix} -q^{*\prime} \\ A \end{pmatrix} \alpha^i$$

Markets clear:

$$\sum_{i\in\mathcal{I}} \alpha^{i*} = 0$$

Clearly any equilibrium price vector must preclude arbitrage for the maximization problem to have a well-defined solution. Indeed, we can infer state prices from

the agents' first order conditions:

$$\pi_s = \frac{\partial u^i\left(x^i\right)}{\partial x_s^i}$$

If $J = S$ (remember the assets dividends are assumed to be linearly independent), then a financial markets equilibrium is equivalent to a Walrasian equilibrium. There will be a unique state-price vector $\pi \in \mathbb{R}_{++}^S$ such that $q = \pi' A$. This will be an equilibrium price vector for a Walrasian economy; the resulting allocations are the same in the two equilibria.

More interesting is the case where $J < S$ so that markets are incomplete. Under our assumptions, a financial markets equilibrium will still exist, but the equilibrium allocation may not be efficient. To see why, let's look at an example.

Suppose there are two states and there is a single bond that pays 1 in each state: $d = (1,1)'$. Suppose there are two agents with endowments:

$$e^1 = (1,2,1)$$
$$e^2 = (1,1,2)$$

and that both agents have identical utility:

$$u^i(x_0, x_1, x_2) = \log\ x_0 + \frac{1}{2}\log\ x_1 + \frac{1}{2}\log\ x_2$$

You can check as an exercise that the unique equilibrium will have no trade in the bond so everyone will just consume there endowment. This allocation, however, is Pareto dominated by the feasible allocation $x^1 = x^2 = (1,1.5,1.5)$.

The first welfare theorem fails because the set of existing securities does not allow the agents to suitably insure themselves against adverse states. There is still as sense, however, in which equilibrium exhausts the gains from trade.

Definition 10 Given endowments $\left(e^i\right)_{i\in\mathcal{I}}$ and assets A, an allocation $\left(x^i\right)_{i\in\mathcal{I}}$ is constrained efficient if $\sum_{i\in\mathcal{I}}\left(x^i - e^i\right) \leq 0, x^i - e^i \in \langle A\rangle$ for all $i \in \mathcal{I}$ and there exists no alternative allocation $\left(\hat{x}^i\right)_{i\in\mathcal{I}}$ that Pareto dominates $\left(x^i\right)_{i\in\mathcal{I}}$ and also satisfies $\sum_{i\in\mathcal{I}}\left(\hat{x}^i - e^i\right) \leq 0$ and $\hat{x}^i - e^i \in \langle A\rangle$ for all $i \in \mathcal{I}$. If you're interested, you can try proving the following weaker welfare theorem:

Theorem 16 If utility functions are strictly increasing, a financial markets equilibrium is constrained efficient.