Solutions Manual for

MICROECONOMIC THEORY

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CHAPTER 1

1.B.1 Since $y \succ z$ implies $y \succsim z$, the transitivity implies that $x \succsim z$. Suppose that $z \succsim x$. Since $y \succsim z$, the transitivity then implies that $y \succsim x$. But this contradicts $x \succ y$. Thus we cannot have $z \succsim x$. Hence $x \succ z$.

1.B.2 By the completeness, $x \succsim x$ for every $x \in X$. Hence there is no $x \in X$ such that $x \succ x$. Suppose that $x \succ y$ and $y \succ z$, then $x \succ y \succsim z$. By (iii) of Proposition 1.B.1, which was proved in Exercise 1.B.1, we have $x \succ z$. Hence $\succ$ is transitive. Property (i) is now proved.

As for (ii), since $x \succsim x$ for every $x \in X$, $x \sim x$ for every $x \in X$ as well. Thus $\sim$ is reflexive. Suppose that $x \sim y$ and $y \sim z$. Then $x \succsim y$, $y \succsim z$, $y \succsim x$, and $z \succsim y$. By the transitivity, this implies that $x \succsim z$ and $z \succsim x$. Thus $x \sim z$. Hence $\sim$ is transitive. Suppose x that $\sim$ y. Then $x \succsim y$ and $y \succsim x$. Thus $y \succsim x$ and $x \succsim y$. Hence $y \sim x$. Thus $\sim$ is symmetric Property (ii) is now proved.

1.B.3 Let $x \in X$ and $y \in X$. Since $u(\cdot)$ represents $\succsim$, $x \succsim y$ if and only if $u(x) \geq u(y)$. Since $f(\cdot)$ is strictly increasing, $u(x) \geq u(y)$ if and only if $v(x) \geq v(y)$. Hence $x \succsim y$ if and only if $v(x) \geq v(y)$. Therefore $v(\cdot)$ represents $\succsim$.

1.B.4 Suppose first that $x \succsim y$. If, furthermore, $y \succsim x$, then $x \sim y$ and hence $u(x) = u(y)$. If, on the contrary, we do not have $y \succsim x$, then $x \succ y$. Hence $u(x) > u(y)$. Thus, if $x \succsim y$, then $u(x) \geq u(y)$.

Suppose conversely that $u(x) \geq u(y)$. If, furthermore, $u(x) = u(y)$, then

$x \sim y$ and hence $x \succsim y$. If, on the contrary, $u(x) > u(y)$, then $x \succ y$, and hence $x \succsim y$. Thus, if $u(x) \geq u(y)$, then $x \succsim y$. So $u(\cdot)$ represents $\succsim$.

1.B.5 First, we shall prove by induction on the number N of the elements of X that, if there is no indifference between any two different elements of X, then there exists a utility function. If $N = 1$, there is nothing to prove: Just assign any number to the unique element. So let $N > 1$ and suppose that the above assertion is true for $N - 1$. We will show that it is still true for N. Write $X = \{x_1,\ldots,x_{N-1},x_N\}$. By the induction hypothesis, $\succsim$ can be represented by a utility function $u(\cdot)$ on the subset $\{x_1,\ldots,x_{N-1}\}$. Without loss of generality we can assume that $u(x_1) > u(x_2) > \ldots > u(x_{N-1})$.

Consider the following three cases:

Case 1: For every $i < N$, $x_N \succ x_i$.

Case 2: For every $i < N$, $x_i \succ x_N$.

Case 3: There exist $i < N$ and $j < N$ such that $x_i \succ x_N \succ x_j$.

Since there is no indifference between two different elements, these three cases are are exhaustive and mutually exclusive. We shall now show how the value of $u(x_N)$ should be determined, in each of the three cases, for $u(\cdot)$ to represent $\succsim$ on the whole X.

If Case 1 applies, then take $u(x_N)$ to be larger than $u(x_1)$. If Case 2 applies, take $u(x_N)$ to be smaller than $u(x_{N-1})$. Suppose now that Case 3 applies. Let $I = \{i \in \{1,\ldots, N - 1\}: x_i \succ x_{N+1}\}$ and $J = \{j \in \{1,\ldots, N - 1\}: x_{N+1} \succ x_j\}$. Completeness and the assumption that there is no indifference implies that $I \cup J = \{1,\ldots, N - 1\}$. The transitivity implies that both I and J are "intervals," in the sense that if $i \in I$ and $i' < i$, then $i' \in I$; and if $j \in J$ and $j' > j$, then $j' \in J$. Let $i^* = \max I$, then $i^* + 1 = \min J$. Take

$u(x_N)$ to lie in the open interval $(u(x_{i^*+1}), u(x_{i^*}))$. Then it is easy to see that $u(\cdot)$ represents $\succsim$ on the whole X.

Suppose next that there may be indifference between some two elements of $X = \{x_1, \dots, x_N\}$. For each n = 1,...,N, define $X_n = \{x_m \in X: x_m \sim x_n\}$. Then, by the reflexivity of ~ (Proposition 1.B.1(ii)), $\cup_{n=1}^{N} X_n = X$. Also, by the transitivity of ~ (Proposition 1.B.1(ii)), if $X_n \neq X_m$, then $X_n \cap X_m \neq \emptyset$. So let M be a subset of {1,...,N} such that $X = \cup_{m \in M} X_m$ and $X_m \neq X_n$ for any $m \in M$ and any $n \in M$ with $m \neq n$. Define an relation $\succsim^*$ on $\{X_m: m \in M\}$ by letting $X_m \succsim^* X_n$ if and only if $x_m \succsim x_n$. In fact, by the definition of M, there is no indifference between two different elements of $\{X_m: m \in M\}$. Thus, by the preceding result, there exists a utility function $u^*(\cdot)$ that represents $\succsim^*$. Then define $u: X \to \mathbb{R}$ by $u(x_n) = u^*(X_m)$ if $m \in M$ and $x_n \in X_m$. It is easy to show that, by the transitivity, $u(\cdot)$ represents $\succsim$.

1.C.1 If $y \in C(\{x,y,z\})$, then the WA would imply that $y \in C(\{x,y\})$. But contradicts the equality $C(\{x,y\}) = \{x\}$. Hence $y \notin C(\{x,y,z\})$. Thus $C(\{x,y,z\}) \in \{\{x\},\{z\},\{x,z\}\}$.

1.C.2 The property in the question are equivalent to the following property: If $B \in \mathcal{B}$, $B' \in \mathcal{B}$, $x \in B$, $y \in B$, $x \in B'$, $y \in B'$, $x \in C(B)$, and $y \in C(B')$, then $x \in C(B')$ and $y \in C(B)$. We shall thus prove the equivalence between this property and the Weak Axiom.

Suppose first that the Weak Axiom is satisfied. Assume that $B \in \mathcal{B}$, $B' \in \mathcal{B}$, $x \in B$, $y \in B$, $x \in B'$, $y \in B'$, $x \in C(B)$, and $y \in C(B')$. If we apply the Weak Axiom twice, we obtain $x \in C(B')$ and $y \in C(B)$. Hence the above property is also satisfied.

Suppose conversely that the above property is satisfied. Let $B \in \mathcal{B}$, $x \in B$, $y \in B$, $x \in B'$, and $x \in C(B)$. Furthermore, let $B' \in \mathcal{B}$, $x \in B'$, $y \in B'$, and $y \in C(B')$. Then the above condition implies that $x \in C(B')$ (and $y \in C(B)$). Thus the Weak Axiom is satisfied.

1.C.3 **(a)** Suppose that $x \succ^* y$, then there is some $B \in \mathcal{B}$ such that $x \in B$, $y \in B$, $x \in C(B)$, and $y \notin C(B)$. Thus $x \succsim^* y$. Suppose that $y \succsim^* x$, then there exists $B \in \mathcal{B}$ such that $x \in B$, $y \in B$ and $x \in C(B)$. But the Weak Axiom implies that $y \in C(B)$, which is a contradiction. Hence if $x \succ^* y$, then we cannot have $y \succsim^* x$. Hence $x \succ^{**} y$.

Conversely, suppose that $x \succ^{**} y$, then $x \succsim^* y$ but not $y \succsim^* x$. Hence there is some $B \in \mathcal{B}$ such that $x \in B$, $y \in B$, $x \in C(B)$ and if $x \in B'$ and $y \in B'$ for any $B' \in \mathcal{B}$, then $y \notin C(B')$. In particular, $x \in C(B)$ and $y \notin C(B)$. Thus $x \succ^* y$.

The equality of the two relation is not guaranteed without the WA. As can be seen from the above proof, the WA is not necessary to guarantee that if $x \succ^{**} y$, then $x \succ^* y$. But the converse need not be true, as shown by the following example. Define $X = \{x,y,z\}$, $\mathcal{B} = \{\{x,y\},\{x,y,z\}\}$, $C(\{x,y\}) = \{x\}$, and $C(\{x,y,z\}) = \{y\}$. Then $x \succ^* y$ and $y \succ^* x$. But neither $x \succ^* y$ nor $y \succ^* x$.

(b) The relation $\succ^*$ need not be transitive, as shown by the following example. Define $X = \{x,y,z\}$, $\mathcal{B} = \{\{x,y\},\{y,z\}\}$, $C(\{x,y\}) = \{x\}$ and $C(\{y,z\}) = \{y\}$. Then $x \succ^* y$ and $y \succ^* z$. But we do not have $x \succsim^* z$ (because neither of the two sets in $\mathcal{B}$ includes $\{x,z\}$) and hence we do not have $x \succ^* z$ either.

(c) According to the proof of Proposition 1.D.2, if $\mathcal{B}$ includes all three-element subset of X, then $\succsim^*$ is transitive. By Proposition 1.B.1(i), $\succ^{**}$ is transitive. Since $\succ^*$ is equal to $\succ^{**}$, $\succ^*$ is also transitive.

An alternative proof is as follows: Let $x \in X$, $y \in X$, $z \in X$, $x \succ^* y$, and $y \succ^* z$. Then $\{x,y,z\} \in \mathcal{B}$ and, by (a), $x \succ^{**} y$, and $y \succ^{**} z$. Hence we have neither $y \succsim^* x$ nor $z \succsim^* y$. Since $\succsim^*$ rationalizes $(\mathcal{B},C(\cdot))$, this implies that $y \notin C(\{x,y,z\})$ and $z \notin C(\{x,y,z\})$. Since $C(\{x,y,z\}) \neq \varnothing$, $C(\{x,y,z\}) = \{x\}$. Thus $x \succ^* z$.

1.D.1 The simplest example is $X = \{x,y\}$, $\mathcal{B} = \{\{x\},\{y\}\}$, $C(\{x\}) = \{x\}$, $C(\{y\}) = \{y\}$. Then any rational preference relation of X rationalizes $C(\cdot)$.

1.D.2 By Exercise 1.B.5, let $u(\cdot)$ be a utility representation of $\succsim$. Since X is finite, for any $B \subset X$ with $B \neq \varnothing$, there exists $x \in B$ such that $u(x) \geq u(y)$ for all $y \in B$. Then $x \in C^*(B,\succsim)$ and hence $C^*(B,\succsim) \neq \varnothing$. (A direct proof with no use of utility representation is possible, but it is essentially the same as the proof of Exercise 1.B.5.)

1.D.3 Suppose that the Weak Axiom holds. If $x \in C(X)$, then $x \in C(\{x,z\})$, which contradicts the equality $C(\{x,z\}) = \{z\}$. If $y \in C(X)$, then $y \in C(\{x,y\})$, which contradicts $C(\{x,y\}) = \{x\}$. If $z \in C(X)$, then $z \in C(\{y,z\})$, which contradicts $C(\{x,z\}) = \{y\}$. Thus $(\mathcal{B},C(\cdot))$ must violate the Weak Axiom.

1.D.4 Let $\succsim$ rationalize $C(\cdot)$ relative to $\mathcal{B}$. Let $x \in C(B_1 \cup B_2)$ and $y \in C(B_1) \cup C(B_2)$, then $x \succsim y$ because $B_1 \cup B_2 \supset C(B_1) \cup C(B_2)$. Thus $x \in C(C(B_1) \cup C(B_2))$.

Let $x \in C(C(B_1) \cup C(B_2))$ and $y \in B_1 \cup B_2$, then there are four cases:

Case 1. $x \in C(B_1)$, $y \in B_1$.

Case 2. $x \in C(B_1)$, $y \in B_2$.

Case 3. $x \in C(B_2)$, $y \in B_1$.

Case 4. $x \in C(B_2)$, $y \in B_2$.

If either Case 1 or 4 is true, then $x \succsim y$ follows directly from rationalizability. If Case 2 is true, then pick any $z \in C(B_2)$. Then $z \succsim y$. Since $x \in C(C(B_1) \cup C(B_2))$, $x \succsim z$. Hence, by the transitivity, $x \succsim y$. If Case 3 is true, then pick any $z \in C(B_1)$ and do the same argument as for Case 2.

1.D.5 **(a)** Assign probability 1/6 to each of the six possible preferences, which are $x \succ y \succ z$, $x \succ z \succ y$, $y \succ x \succ z$, $y \succ z \succ x$, $z \succ x \succ y$, and $z \succ y \succ x$.

(b) If the given stochastic choice function were rationalizable, then the probability that at least one of $x \succ y$, $y \succ z$, and $z \succ x$ holds would be at most $3 \times (1/4) = 3/4$. But, in fact, at least one of the three relations always holds, because, if the first two do not hold, then $y \succ x$ and $z \succ y$. Hence the transitivity implies the third. Thus, the given stochastic choice function is not rationalizable.

(c) The same argument as in (b) can be used to show that $\alpha \geq 1/3$. Since $C(\{x,y\}) = C(\{y,z\}) = C(\{z,x\}) = (\alpha, 1 - \alpha)$ is equivalent to $C(\{y,x\}) = C(\{z,y\}) = C(\{x,z\}) = (1 - \alpha, \alpha)$, if we apply the same argument as in (b) to $y \succ x$, $z \succ y$, and $x \succ z$, then we can establish $1 - \alpha \geq 1/3$, that is, $\alpha \leq 2/3$. Thus, in order for the given stochastic choice function is rationalizable, it is necessary that $\alpha \in [1/3,2/3]$. Moreover, this condition is actually sufficient: For any $\alpha \in [1/3,2/3]$, assign probability $\alpha - 1/3$ to each of $x \succ$

$y \succ z$, $y \succ z \succ x$, and $z \succ x \succ y$; assign probability $2/3 - \alpha$ to each of $x \succ z \succ y$, $y \succ x \succ z$, and $z \succ y \succ x$. Then we obtain the given stochastic choice function.

CHAPTER 2

2.D.1 Let p_2 be the price of the consumption good in period 2, measured in units of the consumption good in period 1. Let x_1, x_2 be the consumption levels in periods 1 and 2, respectively. Then his lifetime Walrasian budget set is equal to $\{x \in \mathbb{R}^2_+ : x_1 + p_2x_2 \le w\}$.

2.D.2 $\{(x,h) \in \mathbb{R}^2_+ : h \le 24,\ px + h \le 24\}$.

2.D.3 **(a)** No. In fact, the budget set consists of the two points, each of which is the intersection of the budget line and an axis.

(b) Let $x \in B_{p,w}$, $x' \in B_{p,w}$, and $\lambda \in [0,1]$. Write $x'' = \lambda x + (1 - \lambda)x'$. Since X is convex, $x'' \in X$. Moreover, $p \cdot x'' = \lambda(p \cdot x) + (1 - \lambda)(p \cdot x') \le \lambda w + (1 - \lambda)w = w$. Thus $x'' \in B_{p,w}$.

2.D.4 It follows from a direct calculation that consumption level M can be attained by $(8 + (M - 8s)/s')$ hours of labor. It follows from the definition that $(24,0)$ and $(16 - (M - 8s)/s', M)$ are in the budget set. But their convex combination of these two consumption vectors with ratio

$$\left(\frac{\frac{M - 8s}{s'}}{8 + \frac{M - 8s}{s'}},\ \frac{8}{8 + \frac{M - 8s}{s'}} \right)$$

is not in the budget set: the amount of leisure of this combination equals to 16 (so the labor is eight hours), but the amount of the consumption good is

$$M \frac{8}{8 + \frac{M - 8s}{s'}} > M \frac{8}{8 + \frac{M - 8s}{s}} = M \frac{8}{M/s} = 8s.$$

2.E.1 The homogeneity can be checked as follows:

$$x_1(\alpha p,\alpha w) = \frac{\alpha p_2}{\alpha p_1 + \alpha p_2 + \alpha p_3}\frac{\alpha w}{\alpha p_1} = \frac{p_2}{p_1 + p_2 + p_3}\frac{w}{p_1} = x_1(p,w),$$

$$x_2(\alpha p,\alpha w) = \frac{\alpha p_3}{\alpha p_1 + \alpha p_2 + \alpha p_3}\frac{\alpha w}{\alpha p_2} = \frac{p_3}{p_1 + p_2 + p_3}\frac{w}{p_2} = x_2(p,w),$$

$$x_3(\alpha p,\alpha w) = \frac{\alpha p_1}{\alpha p_1 + \alpha p_2 + \alpha p_3}\frac{\alpha w}{\alpha p_3} = \frac{p_1}{p_1 + p_2 + p_3}\frac{w}{p_3} = x_3(p,w).$$

To see if the demand function satisfies Walras' law, note that

$$p\cdot x(p,w) = \frac{\beta p_1 + p_2 + p_3}{p_1 + p_2 + p_3}w.$$

Hence $p\cdot x(p,w) = w$ if and only if $\beta = 1$. Therefore the demand function satisfies Walras' law if and only if $\beta = 1$.

2.E.2 Multiply by p_k/w both sides of (2.E.4), then we obtain

$$\textstyle\sum_{\ell=1}^{L}(p_\ell x_\ell(p,w)/w)(\partial x_\ell(p,w)/\partial p_k)(p_k/x_\ell(p,w)) + p_k x_k(p,w)/w = 0.$$

Hence $\sum_{\ell=1}^{L} b_\ell(p,w)\varepsilon_{\ell k}(p,w) + b_k(p,w) = 0$.

By (2.E.6), $\sum_{\ell=1}^{L}(p_\ell x_\ell(p,w)/w)(\partial x_\ell(p,w)/\partial w)(w/x_\ell(p,w)) = 1$. Hence $\sum_{\ell=1}^{L} b_\ell(p,w)\varepsilon_{\ell w}(p,w) = 1$.

2.E.3. There are two ways to verify that $p\cdot D_p x(p,w)p = -w$.

One way is to post-multiply (2.E.5) by p, then $p\cdot D_p x(p,w)\, p + w = 0$ by Walras' law.

The other way is to pre-multiply (2.E.1) by p^T, then $p\cdot D_p x(p,w)p + p\cdot D_w x(p,w)w = 0$. By Proposition 2.E.3, this is equal to $p\cdot D_p x(p,w)p + w = 0$.

An interpretation is that, when all prices are doubled, in order for the consumer to stay at the same consumption, it is necessary to increase his wealth by w.

2.E.4 By differentiating the equation $x(p,\alpha w) = \alpha x(p,w)$ with respect to α and evaluating at $\alpha = 1$, we obtain $wD_w x(p,w) = x(p,w)$. Hence $D_w x(p,w) = (1/w)x(p,w)$. Hence $\varepsilon_{\ell w} = (\partial x_\ell(p,w)/\partial w)(w/x_\ell(p,w)) = 1$. This means that an one-percent increase in wealth will increase the consumption level for all goods by one percent.

Since $(1/w)x(p,w) = x(p,1)$ by the homogeneity assumption, $D_w x(p,w)$ is a function of p only. The assumption also implies that the wealth expansion path, $E_p = \{x(p,w)\colon w > 0\}$, is a ray going through $x(p,1)$.

2.E.5 Since $x(p,w)$ is homogeneous of degree one with respect to w, $x(p,\alpha w) = \alpha x(p,w)$ for every $\alpha > 0$. Thus $x_\ell(p,w) = x_\ell(p,1)w$. Since $\partial x_\ell(p,1)/\partial p_k = \partial \varphi_\ell(p)/\partial p_k = 0$ whenever $k \neq \ell$, $x_\ell(p,1)$ is actually a function of p_ℓ alone. So we can write $x_\ell(p,w) = x_\ell(p_\ell)$. Since $x(p,w)$ is homogeneous of degree zero, $x_\ell(p_\ell)$ must be homogeneous of degree - 1 (in p_ℓ). Hence there exists $\alpha_\ell > 0$ such that $x_\ell(p_\ell) = \alpha_\ell/p_\ell$. By Walras' law, $\sum_\ell p_\ell(\alpha_\ell/p_\ell)w = w\sum_\ell \alpha_\ell = w$. We must thus have $\sum_\ell \alpha_\ell = 1$.

2.E.6 When $\alpha = 1$, Walras' law and homogeneity hold. Hence the conclusions of Propositions 2.E.1 - 2.E.3 hold.

2.E.7 By Walras' law,

$$x_2 = (w - p_1x_1)/p_2 = w/p_2 - (p_1/p_2)(\alpha w/p_1) = (1 - \alpha)w/p_2.$$

This demand function is thus homogeneous of degree zero.

2.E.8 For the first part, note that

$$\ln x_\ell(p,w) = \ln x_\ell(\exp(\ln p_1),\ldots,\exp(\ln p_L),\exp(\ln w)),$$

Thus, by the chain rule,

$$\frac{d(\ln x_\ell(p,w))}{d(\ln p_k)} = \frac{\frac{\partial x_\ell}{\partial p_k}(p,w)\cdot\exp(\ln p_k)}{x_\ell(p,w)} = \frac{\frac{\partial x_\ell}{\partial p_k}(p,w)\cdot p_k}{x_\ell(p,w)} = \varepsilon_{\ell k}(p,w).$$

Similarly,

$$\frac{d(\ln x_\ell(p,w))}{d(\ln w)} = \frac{\frac{\partial x_\ell}{\partial w}(p,w)\cdot\exp(\ln w)}{x_\ell(p,w)} = \frac{\frac{\partial x_\ell}{\partial w}(p,w)w}{x_\ell(p,w)} = \varepsilon_{\ell w}(p,w).$$

Since $\alpha_1 = d(\ln x_\ell(p,w))/d(\ln p_1)$, $\alpha_2 = d(\ln x_\ell(p,w))/d(\ln p_2)$, and $\alpha_3 = d(\ln x_\ell(p,w))/d(\ln w)$, the assertion is established.

2.F.1 We proved in Exercise 1.C.2 that Definition 1.C.1 and the property in the exercise is equivalent. It is easy to see that the latter is equivalent to the following property: For every $B \in \mathcal{B}$ and $B' \in \mathcal{B}$, if $C(B) \cap B' \neq \varnothing$ and $B \cap C(B') \neq \varnothing$, then $C(B) \cap B' \subset C(B')$ and $B \cap C(B') \subset C(B)$. If $C(\cdot)$ is single-valued, then this property is equivalent to the following one: For every $B \in \mathcal{B}$ and $B' \in B$, if $C(B) \subset B'$ and $B \subset C(B')$, then $C(B) = C(B')$. In the context of Walrasian demand functions, this can be restated as follows: For any (p,w) and (p',w'), if $p\cdot x(p',w') \leq w$ and $p'\cdot x(p,w) \leq w'$, then $x(p,w) = x(p',w')$. But this is the contraposition of the property stated in Definition 2.F.1. Hence Definitions 1.C.1 and 2.F.1 are equivalent.

2.F.2 It is straightforward to check that the Weak Axiom holds. In fact, if $p^i\cdot x^j \leq 8$ and $i \neq j$, then $p^j\cdot x^i = 9$. Since $p^2\cdot x^1 = 8$, x^2 is revealed preferred to x^1. Similarly, since $p^1\cdot x^3 = 8$, x^1 is revealed preferred to x^3.

But, since $p^3 \cdot x^2 = 8$, x^3 is revealed preferred to x^2.

2.F.3 [First printing errata: Add the sentence "Assume that the weak axiom is satisfied." in (b) and (c).] Denote the demand for good 2 in year 2 by y.

(a) His behavior violates the weak axiom if

$$100 \cdot 120 + 100y \le 100 \cdot 100 + 100 \cdot 100$$

and

$$100 \cdot 100 + 80 \cdot 100 \le 100 \cdot 120 + 80y.$$

That is, the Weak Axiom is violated if $y \in [75,80]$.

(b) The bundle in year 1 is revealed preferred if

$$100 \cdot 120 + 100y \le 100 \cdot 100 + 100 \cdot 100$$

and

$$100 \cdot 100 + 80 \cdot 100 > 100 \cdot 120 + 80y,$$

that is, $y < 75$.

(c) The bundle in year 2 is revealed preferred if

$$100 \cdot 100 + 80 \cdot 100 \le 100 \cdot 120 + 80y$$

and

$$100 \cdot 120 + 100y > 100 \cdot 100 + 100 \cdot 100,$$

that is, $y > 80$.

(d) For any value of y, we have sufficient information to justify exactly one of (a), (b), and (c).

(e) We shall prove that if $y < 75$, then good 1 is an inferior good. So suppose that $y < 75$. Then

$$100\cdot 120 + 100y \le 100\cdot 100 + 100\cdot 100$$

and

$$100\cdot 100 + 80\cdot 100 > 100\cdot 120 + 80y.$$

Hence the real wealth decreases from year 1 to 2. Also the relative price of good 1 increases. But the demand for good 2, y, decreases because $y < 75 < 100$. This means that the wealth effect on good 1 must be negative. Hence it is an inferior good.

(f) We shall prove that if $80 < y < 100$, then good 2 is an inferior good. So suppose that $80 < y < 100$. Then

$$100\cdot 100 + 80\cdot 100 \le 100\cdot 120 + 80y$$

and

$$100\cdot 120 + 100y > 100\cdot 100 + 100\cdot 100.$$

Hence the real wealth increases from year 1 to 2. Also the relative price of good 2 decreases. But the demand for good 2, y, decreases because $y < 100$. This means that the wealth effect on good 2 must be negative. Hence it is an inferior good.

2.F.4 **(a)** If $L_Q < 1$, then $(p_0\cdot x_1)/(p_0\cdot x_0) < 1$ and hence $p_0\cdot x_1 < p_0\cdot x_0$. Thus the consumer has a revealed preference for x_0 over x_1.

(b) If $P_Q > 1$, then $(p_1\cdot x_1)/(p_1\cdot x_0) > 1$ and hence $p_1\cdot x_1 > p_1\cdot x_0$. Thus the consumer has a revealed preference for x_1 over x_0.

(c) If $p_2 = \lambda p_1$ and $w_2 = \lambda w_1$, and $x_1 = x_2$, then $E_Q = \lambda$. Hence, by taking λ larger or smaller than one, we can make E_Q larger or smaller than one. But this obviously does not have any revealed preference relationship.

2.F.5 We shall first prove the discrete version. By the homogeneity of degree one with respect to wealth, it is enough to show that

$$(p' - p)\cdot(x(p',1) - x(p,1)) \le 0 \text{ for every p and p'.}$$

Since

$$x(p',1) - x(p,1) = \frac{1}{p'\cdot x(p,1)}(x(p',p'\cdot x(p,1)) - x(p,1)) + (x(p, \frac{1}{p'\cdot x(p,1)}) - x(p,1)),$$

it is sufficient to show that

$$(p' - p)\cdot(x(p',p'\cdot x(p,1)) - x(p,1)) \le 0,$$

and

$$(p' - p)\cdot(x(p, \frac{1}{p'\cdot x(p,1)}) - x(p,1)) \le 0.$$

For the first inequality, note that

$$(p' - p)\cdot(x(p',p'\cdot x(p,1)) - x(p,1)) = - p\cdot x(p',p'\cdot x(p,1)) + 1.$$

If $x(p',p'\cdot x(p,1)) = x(p,1)$, then the value is equal to zero. If $x(p',p'\cdot x(p,1)) \ne x(p,1)$, then the weak axiom implies that $p\cdot x(p',p'\cdot x(p,1)) > 1$. Hence the above value is negative.

As for the second inequality,

$$\begin{aligned}
&(p' - p)\cdot(x(p, \frac{1}{p'\cdot x(p,1)}) - x(p,1)) \\
&= p'\cdot x(p, \frac{1}{p'\cdot x(p,1)}) - p'\cdot x(p,1) - \frac{1}{p'\cdot x(p,1)} + 1. \\
&= 2 - (p'\cdot x(p,1) + \frac{1}{p'\cdot x(p,1)}) \\
&\le 2 - 2\sqrt{(p'\cdot x(p,1))(\frac{1}{p'\cdot x(p,1)})} \\
&= 2 - 2 = 0.
\end{aligned}$$

The infinitesimal version goes as follows. By differentiating $x(p,\alpha w) = \alpha x(p,w)$ with respect to α and evaluating at $\alpha = 1$, we obtain $D_w x(p,w)w = x(p,w)$. Hence

$$S(p,w) = D_p x(p,w) + D_w x(p,w) x(p,w)^T = D_p x(p,w) + (1/w) x(p,w) x(p,w)^T.$$

Thus

$$D_p x(p,w) = S(p,w) - (1/w) x(p,w) x(p,w)^T.$$

By Proposition 2.F.2, $S(p,w)$ is negative semidefinite. Moreover, since $v \cdot (x(p,w)x(p,w)^T)v = -(v \cdot x(p,w))^2$, the matrix $-(1/w)x(p,w)x(p,w)^T$ is also negative semidefinite. Thus $D_p x(p,w)$ is negative semidefinite.

2.F.6 Clearly the weak axiom implies that there exists $w > 0$ such that for every p, p', and w', if $p \cdot x(p',w') \le w$ and $x(p',w') \ne x(p,w)$, then $p' \cdot x(p,w) > w$.

Conversely, suppose that such a $w > 0$ exists and that $p \cdot x(p',w') \le w$ and $x(p',w') \ne x(p,w)$. Let $\alpha = w'/w$. Then $x(p',w') = x(p',\alpha w) = x(\alpha^{-1}p',w)$ by the homogeneity assumption, and $p \cdot x(\alpha^{-1}p',w) \le w$ and $x(\alpha^{-1}p',w) \ne x(p,w)$. But this implies that $(\alpha^{-1}p') \cdot x(p,w) > w$, or, equivalently, $p' \cdot x(p,w) > \alpha w = w'$. Thus the weak axiom holds.

2.F.7 By Propositions 2.E.2 and 2.E.3,

$$p \cdot S(p,w) = p \cdot D_p x(p,w) + p \cdot D_w x(p,w) x(p,w)^T = p \cdot D_p x(p,w) + x(p,w)^T = 0$$

By Proposition 2.E.1 and Walras' law,

$$S(p,w)p = D_p x(p,w)p + D_w x(p,w) x(p,w)^T p = D_p x(p,w)p + D_w x(p,w) w = 0.$$

2.F.8
$$\begin{aligned}\hat{s}_{\ell k}(p,w) &= \frac{p_k}{x_\ell(p,w)} s_{\ell k}(p,w) \\ &= \frac{p_k}{x_\ell(p,w)} \frac{\partial x_\ell}{\partial p_k}(p,w) + \frac{p_k}{x_\ell(p,w)} \frac{\partial x_\ell}{\partial w}(p,w) x_k(p,w) \\ &= \varepsilon_{\ell k}(p,w) + \frac{w}{x_\ell(p,w)} \frac{\partial x_\ell}{\partial w}(p,w) \frac{p_k x_k(p,w)}{w} \\ &= \varepsilon_{\ell k}(p,w) + \varepsilon_{\ell w}(p,w) b_k(p,w).\end{aligned}$$

2.F.9 **(a)** Since $x^TA^Tx = (x^TAx)^T = x^TAx$, a matrix A is negative definite if and only if $x^TAx + x^TA^Tx < 0$ for every $x \in \mathbb{R}^n\setminus\{0\}$. Since $x^TAx + x^TA^Tx = x^T(A + A^T)x$, this is equivalent to the negative definiteness of $A + A^T$. Thus A is negative definite if and only if so is $A + A^T$. The case of negative definiteness can be proved similarly.

The following examples shows that the determinant condition is not sufficient for the nonsymmetric case. Let $A = \begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix}$, then $A_{11} = -1$ and $A_{22} = 1$. But $(1, 1)\begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$. Hence A is not negative semidefinite.

(b) Let $S(p,w)$ be a substitution matrix. By Proposition 2.F.3, $S(p,w)p = 0$ and hence $s_{12}(p,w) = (-p_1/p_2)s_{11}(p,w)$. Also $p\cdot S(p,w) = 0$ and hence $s_{21}(p,w) = (-p_1/p_2)s_{11}(p,w)$. Thus $s_{22}(p,w) = (p_1^2/p_2^2)s_{11}(p,w)$. Thus, for every $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$,

(*) $$\begin{aligned} v\cdot S(p,w)v &= s_{11}(p,w)(v_1^2 - (2p_1/p_2)v_1v_2 + (p_1^2/p_2^2)v_2^2) \\ &= s_{11}(p,w)(v_1 - (p_1/p_2)v_2)^2. \end{aligned}$$

Now, suppose that $S(p,w)$ is negative semidefinite and of rank one. According to (*), the negative semidefiniteness implies that $s_{11}(p,w) \le 0$. Being of rank one implies that $s_{11}(p,w) \ne 0$. Hence $s_{11}(p,w) < 0$. Thus $s_{22}(p,w) < 0$. Conversely, let $s_{11}(p,w) < 0$, then, by (*), $v\cdot S(p,w)v \le 0$ for every v.

2.F.10 **(a)** If $p = (1,1,1)$ and $w = 1$, then, by a straightforward calculation, we obtain

$$S(p,w) = (1/3)\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Hence $S(p,w)$ is not symmetric. Note that

$$(v_1, v_2)\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -v_1^2 + v_1v_2 - v_2^2 = -(v_1 - v_2/2)^2 - 3v_2^2/4.$$

Hence $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ is negative definite. Thus, by Proposition 2.F.3 and Theorem M.D.4(iii), $S(p,w)$ is negative semidefinite.

(b) Let $p = (1,1,\varepsilon)$ and $w = 1$. Let $\hat{S}(p,w)$ be the 2×2 submatrix of $S(p,w)$ obtained by deleting the last row and column. By a straightforward calculation, we obtain

$$\hat{S}(p,w) = (2+\varepsilon)^{-2}\begin{bmatrix} -2-\varepsilon & 1+2\varepsilon \\ 0 & -3\varepsilon \end{bmatrix}.$$

Thus,

$$(1, 4, 0)S(p,w)\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} = (1, 4)\hat{S}(p,w)\begin{bmatrix} 1 \\ 4 \end{bmatrix} = (2+\varepsilon)^{-2}(2 - 41\varepsilon) > 0,$$

if $\varepsilon > 0$ is sufficiently small. Then $S(p,w)$ is not negative semidefinite and hence the demand function in Exercise 2.E.1 does not satisfy the Weak Axiom.

2.F.11 By Proposition 2.F.3, $S(p,w)p = 0$ and hence $s_{12}(p,w) = (-p_1/p_2)s_{11}(p,w)$. Also $p \cdot S(p,w) = 0$ and hence $s_{21}(p,w) = (-p_1/p_2)s_{11}(p,w)$. (We saw this in the answer for Exercise 2.F.9 as well.) Thus $s_{12}(p,w) = s_{21}(p,w)$.

2.F.12 By applying Proposition 1.D.1 to the Walrasian choice structure, we know that $x(p,w)$ satisfies the weak axiom in the sense of Definition 1.C.1. By Exercise 2.F.1, this implies that $x(p,w)$ satisfies the weak axiom in the sense of Definition 2.F.1.

2.F.13 [First printing errata: In the last part of condition (*) of (b), the inequality $p \cdot x > w$ should be $p' \cdot x > w'$. Also, in the last part of (c), the relation $x' \in x(p,w)$ should be $x' \notin x(p,w)$.]

(a) We say that a Walrasian demand correspondence satisfies the weak axiom if the following condition is satisfied: For any (p,w) and (p',w'), if $x \in x(p,w)$, $x' \in x(p',w')$, $p' \cdot x \le w'$, and $p \cdot x' \le w$, then $x' \in x(p,w)$. Or equivalently, for any (p,w) and (p',w'), if $x \in x(p,w)$, $x' \in x(p',w')$, $p \cdot x' \le w$, and $x' \notin x(p,w)$, then $p' \cdot x > w'$.

(b) If $x \in x(p,w)$, $x' \in x(p',w')$, and $p \cdot x' < w$, then $x' \notin x(p,w)$ by Walras' law. Thus $p' \cdot x > w'$.

(c) If $x \in x(p,w)$, $x' \in x(p',w')$, and $p' \cdot x = w'$, then $(p' - p) \cdot (x' - x) = w - p \cdot x'$. If, furthermore, $x' \in x(p,w)$, then Walras' law implies that $p \cdot x' = w$. Hence $(p' - p) \cdot (x' - x) = 0$. If, on the contrary, $x' \notin x(p,w)$, then the generalized weak axiom implies that $p \cdot x' > w$. Hence $(p' - p) \cdot (x' - x) < 0$.

(d) It can be shown in the same way as in the small-type discussion of the proof of Proposition 2.F.1 that, in order to verify the assertion, it is sufficient to show that the generalized weak axiom holds for all compensated price changes. So suppose that $x \in x(p,w)$, $x' \in x(p',w')$, $p' \cdot x = w'$, and $p \cdot x' \le w$. Then $(p' - p) \cdot (x' - x) = w - p \cdot x' \ge 0$. Hence, by the generalized compensated law of Demand, we must have $(p' - p) \cdot (x' - x) = 0$ and $x' \in x(p,w)$.

2.F.14 Let $p \gg 0$, $w \ge 0$, and $\alpha > 0$. Since $p \cdot x(p,w) \le w$ and $(\alpha p) \cdot x(\alpha p, \alpha w) \le \alpha w$, we have $\alpha p \cdot x(p,w) \le \alpha w$ and $p \cdot x(\alpha p, \alpha w) \le w$. The weak axiom now implies that $x(p,w) = x(\alpha p, \alpha w)$.

2.F.15 Since $\partial x_\ell(p,w)/\partial w = 0$ for both $\ell = 1,2$, we have $s_{\ell k}(p,w) = \partial x_\ell(p,w)/\partial p_k$ for both $\ell = 1,2$ and $k = 1,2$. Hence, let $\hat{S}(p,w)$ be the 2×2 submatrix of $S(p,w)$ obtained by deleting the last row and column, then $\hat{S}(p,w) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. This matrix is negative definite because

$$(v_1, v_2)\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -v_1^2 + v_1 v_2 - v_2^2 = -(v_1 - v_2/2)^2 - 3v_2^2/4.$$

(We saw this in the answer to Exercise 2.F.10(a).) Hence, by Theorem M.D.4(iii), $v \cdot S(p,w)v < 0$ for all v not proportional to p. Since $\hat{S}(p,w)$ is not symmetric, $S(p,w)$ is not symmetric either.

2.F.16 **(a)** The homogeneity can be checked as follows:

$$x_1(\alpha p, \alpha w) = \alpha p_2/\alpha p_3 = p_2/p_3 = x_1(p,w),$$
$$x_2(\alpha p, \alpha w) = -\alpha p_1/\alpha p_3 = -p_1/p_3 = x_2(p,w),$$
$$x_3(\alpha p, \alpha w) = \alpha w/\alpha p_3 = w/p_3 = x_3(p,w).$$

As for Walras' law,

$$p_1 x_1(p,w) + p_2 x_2(p,w) + p_3 x_3(p,w) = (p_1 p_2 - p_2 p_1 + p_3 w)/p_3 = w.$$

(b) Let $p = (1,2,1)$, $w = 1$, $p' = (1,1,1)$, and $w' = 2$, then $x(p,w) = (2, -1, 1)$ and $x(p',w') = (1, -1, 2)$. Thus $p' \cdot x(p,w) = 2 = w'$ and $p \cdot x(p',w') = 1 = w$. Hence the Weak Axiom is violated.

(c) Denote by $D\hat{x}(p,w)$ the 2×2 submatrix of the Jacobian matrix $Dx(p,w)$ obtained by deleting the last row and column, then

$$D\hat{x}(p,w) = (1/p_3)\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Let $\hat{S}(p,w)$ be the 2×2 submatrix of $S(p,w)$ obtained by deleting the last row and column, then $\hat{S}(p,w) = D\hat{x}(p,w) = (1/p_3)\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$, because $\partial x_1(p,w)/\partial w = \partial x_2(p,w)/\partial w = 0$. Note that $\hat{v}\cdot\hat{S}(p,w)\hat{v} = 0$ for every $\hat{v} \in \mathbb{R}^2$. Now let $v \in \mathbb{R}^3$. Note that $v = (v - (v_3/p_3)p) + (v_3/p_3)p$ and the third coordinate of $v - (v_3/p_3)p$ is equal to zero. So denote its first two coordinates by $\hat{v} \in \mathbb{R}^2$. Then, by Proposition 2.F.3, $v\cdot S(p,w)v = \hat{v}\cdot\hat{S}(p,w)\hat{v} = 0$.

2.F.17 **(a)** Yes. In fact, $x_k(\alpha p,\alpha w) = \alpha w/(\sum_\ell \alpha p_\ell) = w/(\sum_\ell p_\ell) = x_k(p,w)$.

(b) Yes. In fact, $p\cdot x(p,w) = \sum_k p_k x_k(p,w) = \sum_k p_k w/(\sum_\ell p_\ell) = w$.

(c) Suppose that $p'\cdot x(p,w) \le w'$ and $p\cdot x(p',w') \le w$. The first inequality implies that $(\sum_\ell p'_\ell)w/(\sum_\ell p_\ell) \le w'$, that is, $w/(\sum_\ell p_\ell) \le w'/(\sum_\ell p'_\ell)$. The second inequality implies similarly that $(\sum_\ell p_\ell)w'/(\sum_\ell p'_\ell) \le w$, that is, $w'/(\sum_\ell p'_\ell) \le w/(\sum_\ell p_\ell)$. Therefore $w/(\sum_\ell p_\ell) = w'/(\sum_\ell p'_\ell)$. Hence $x(p,w) = x(p',w')$. Thus the weak axiom holds.

(d) By calculation, we obtain

$$D_p x(p,w) = (-\,w/(\textstyle\sum_\ell p_\ell)^2)\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix},$$

$$D_w x(p,w) = (1/\textstyle\sum_\ell p_\ell)\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = x(p,w).$$

Hence $S(p,w) = 0$. It is symmetric, negative semidefinite, but not negative definite.

CHAPTER 3

3.B.1 **(a)** Assume that $\succsim$ is strongly monotone and $x \gg y$. Then $x \geq y$ and $x \neq y$. Hence $x \succ y$. Thus $\succsim$ is monotone.

(b) Assume that $\succsim$ is monotone, $x \in X$, and $\varepsilon > 0$. Let $e = (1,\ldots,1) \in \mathbb{R}^L$ and $y = x + (\varepsilon/\sqrt{L})e$. Then $\|y - x\| \leq \varepsilon$ and $y \succ x$. Thus $\succsim$ is locally nonsatiated.

3.B.2 Suppose that $x \gg y$. Define $\varepsilon = \text{Min}\,\{x_1 - y_1, \ldots, x_L - y_L\} > 0$, then, for every $z \in X$, if $\|y - z\| < \varepsilon$, then $x \gg z$. By the local nonsatiation, there exists $z^* \in X$ such that $\|y - z^*\| < \varepsilon$ and $z^* \succ y$. By $x \gg z^*$ and the weak monotonicity, $x \succsim z^*$. By Proposition 1.B.1(iii) (which is implied by the transitivity), $x \succ y$. Thus $\succsim$ is monotone.

3.B.3 Following is an example of a convex, locally nonsatiated preference relation that is not monotone in $\mathbb{R}^2_+$. For example, $x \gg y$ but $y \succ x$.

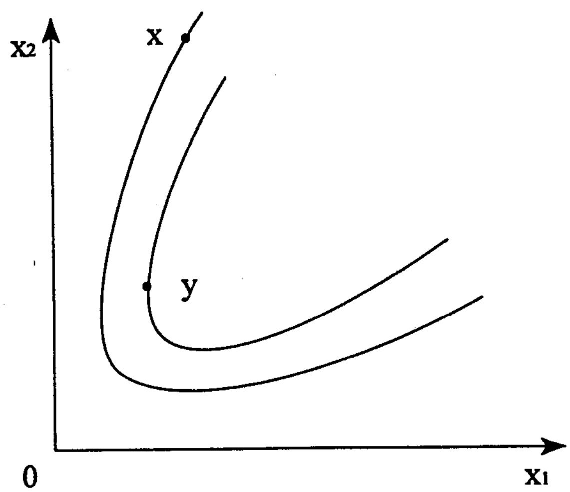


Figure 3.B.3

3.C.1 Let $\succsim$ be a lexicographic ordering. To prove the completeness, suppose that we do not have $x \succsim y$. Then "$y_1 \geq x_1$" and "$x_1 \neq y_1$ or $y_2 > x_2$". Hence either "$y_1 > x_1$" or "$y_1 \geq x_1$ and $y_2 > x_2$". Thus $y \succsim x$.

To prove the transitivity, suppose that $x \succsim y$ and $y \succsim z$. Then $x_1 \geq y_1$ and $y_1 \geq z_1$. Hence $x_1 \geq z_1$. If $x_1 > z_1$, then $x \succsim z$. If $x_1 = z_1$, then $x_1 = y_1 = z_1$. Thus $x_2 \geq y_2$ and $y_2 \geq z_2$. Hence $x_2 \geq z_2$. Thus $x \succsim z$.

To show that the strong monotonicity, suppose that $x \geq y$ and $y \neq x$. This implies either that $x_1 > y_1$ and $x_2 \geq y_2$, or that $x_1 = y_1$ and $x_2 > y_2$. In either case $x \succ y$.

To show the strict convexity, suppose that $y \succsim x$, $z \succsim x$, $y \neq z$, and $\alpha \in (0,1)$. Without loss of generality, assume that $x \neq y$. By the definition of the lexicographic ordering, we have either "$y_1 > x_1$" or "$y_1 = x_1$ and $y_2 > x_2$". On the other hand, since $z \succsim x$, we have either "$z_1 > x_1$" or "$z_1 = x_1$ and $x_2 \geq y_2$." Hence, we have either "$\alpha y_1 + (1 - \alpha)z_1 > x_1$," or "$\alpha y_1 + (1 - \alpha)z_1 = x_1$ and $\alpha y_2 + (1 - \alpha)z_2 > x_2$." Thus $\alpha y + (1 - \alpha)z \succ x$.

3.C.2 Take a sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ such that $x^n \succsim y^n$ for all n, $x^n \to x$, and $y^n \to y$. Then $u(x^n) \geq u(y^n)$ for all n, and the continuity of $u(\cdot)$ implies that $u(x) \geq u(y)$. Hence $x \succsim y$. Thus $\succsim$ is continuous.

3.C.3 One way to prove the assertion is to assume that $\succsim$ is monotone and notice that the proof actually make use only of the closedness of upper and lower contour sets. Then the proposition is applicable to $\succsim$, implying that it has a continuous utility function. Thus, by Exercise 3.C.2, $\succsim$ is continuous.

A more direct proof (without assuming monotonicity or using a utility function) goes as follows. Suppose that there exist two sequences $\{x^n\}$ and

$\{y^n\}$ in X such that $x^n \succsim y^n$ for every n, $x^n \to x \in X$, $y^n \to y \in X$, and $y \succ x$. Since $\{z: y \succ z\}$ is open, there exists a positive integer N_1 such that $y \succ x^n$ for every $n > N_1$. Since $\{z: z \succ x\}$ is open there exists a positive integer N_2 such that $y^n \succ x$ for every $n > N_2$. Conceivably, there are two cases on the sequence $\{y^n\}$:

Case 1: There exists a positive integer N_3 such that $y^n \succsim y$ for every $n > N_3$.

Case 2: There exists a subsequence $(y^{k(n)})$ such that $y \succ y^{k(n)}$ for every n.

If Case 1 applies, then, by Proposition 1.B.1(iii), we have $y^n \succ x^n$ for every $n > \text{Max}\,\{N_1, N_3\}$. This is a contradiction. If Case 2 applies, then there exists a positive integer m such that $k(m) > N_2$. Since $\{z: z \succ y^{k(m)}\}$ is open, there exists a positive integer N_4 such that $y^n \succ y^{k(m)}$ for every $n > N_4$. By $x^n \succsim y^n$ and Proposition 1.B.1(iii), $x^n \succ y^{k(m)}$ for every $n > N_4$. Since $\{z: z \succsim y^{k(m)}\}$ is closed, $x \succsim y^{k(m)}$. But, since $k(m) > N_2$, this is a contradiction.

3.C.4 We provide two examples. The first one is simpler, but the second one satisfies monotonicity, which the first does not.

Example 1. Let $X = \mathbb{R}_+$ and define $u(\cdot): \mathbb{R}_+ \to \mathbb{R}$ by letting $u(x) = 0$ for $x < 1$, $u(x) = 1$ for $x > 1$, and $u(1)$ be any number in $[0,1]$. Denote by $\succsim$ the preference relation represented by $u(\cdot)$. We shall now prove that $\succsim$ is not continuous. In fact, if $u(1) > 0$, then consider a sequence $\{x^n\}$ with $x^n = 1 - 1/n$ for every n. Although $x^n \sim 0$ for every n and $x^n \to 1$, we have $1 \succ 0$. If $u(1) < 1$, then consider a a sequence $\{x^n\}$ with $x^n = 1 + 1/n$ for every n. Although $x^n \sim 2$ for every n and $x^n \to 1$, we have $2 \succ 1$. Note that if $u(x) = 0$, then all lower contour sets are closed. If $u(1) = 1$, then all upper contour sets are closed.

Example 2. Take $X = \mathbb{R}^2_+$ and define a utility function $u(\cdot)$: $\mathbb{R}^2_+ \to \mathbb{R}$ by the following rule:

Case 1. If $x_1 + x_2 \le 2$ and $x \ne (1,1)$, then $u(x) = x_1 + x_2$.

Case 2. If $\min\{x_1,x_2\} \ge 1$ and $x \ne (1,1)$, then $u(x) = \min\{x_1,x_2\} + 2$.

Case 3. If $x_1 + x_2 > 2$, $\min\{x_1,x_2\} < 1$, and $x_1 > x_2$, then

$$u(x) = 3 - (1 - x_2)/(x_1 - 1).$$

Case 4. If $x_1 + x_2 > 2$, $\min\{x_1,x_2\} < 1$, and $x_1 < x_2$, then

$$u(x) = 3 - (1 - x_1)/(x_2 - 1).$$

Case 5. $u(1,1) \in [2,3]$.

The indifference curves of the preference relation $\succsim$ represented by $u(\cdot)$ are described in the following picture:

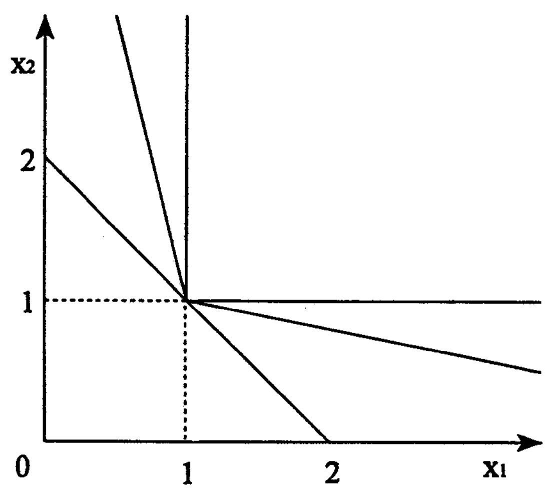


Figure 3.C.4

It follows from this construction that $u(\cdot)$ is continuous at every $x \ne (1,1)$. The preference $\succsim$ is convex and monotone. But, whatever the choice of the value of $u(1,1)$ is, it cannot be continuous at (1,1). In fact,

$(1 - 1/n,\ 1 - 1/n) \to (1,1)$ and $(1 + 1/n,\ 1 + 1/n) \to (1,1)$ as $n \to \infty$, and

$$u(1 - 1/n,\ 1 - 1/n) = 2 - 2/n \to 2;$$

$$u(1 + 1/n,\ 1 + 1/n) = 1 + 1/n \ + 2 \to 3.$$

Hence, if $2 < u(1,1)$, then $(2,0) \succsim (1 - 1/n,\ 1 - 1/n)$ but $(1,1) \succ (2,0)$; if $u(1,1) < 3$, then $(1 + 1/n,\ 1 + 1/n) \succsim (2,1)$ but $(2,1) \succ (1,1)$. If $u(1,1) = 3$, then all upper contour sets of $\succsim$ are closed; if $u(1,1) = 2$, then all lower contour sets of $\succsim$ are closed.

3.C.5 **(a)** Suppose first that $u(\cdot)$ is homogeneous of degree one and let $\alpha \geq 0$, $x \in \mathbb{R}^L_+$, $y \in \mathbb{R}^L_+$, and $x \sim y$. Then $u(x) = u(y)$ and hence $\alpha u(x) = \alpha u(y)$. By the homogeneity, $u(\alpha x) = u(\alpha y)$. Thus $\alpha x \sim \alpha y$.

Suppose conversely that $\succsim$ is homothetic. We shall prove that the utility function constructed in the proof of Proposition 3.C.1 is homogeneous of degree one. Let $x \in \mathbb{R}^L_+$ and $\alpha > 0$, then $u(x)e \sim x$ and $u(\alpha x)e \sim \alpha x$. Since $\succsim$ is homothetic, $\alpha u(x)e \sim \alpha x$. By the transitivity of $\sim$ (Proposition 1.B.1(ii)), $u(\alpha x)e \sim \alpha u(x)e$. Thus $u(\alpha x) = \alpha u(x)$.

(b) Suppose first that $\succsim$ is represented by a utility function of the form $u(x) = x_1 + \phi(x_2,\ldots,x_L)$. Let $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^L_+$, $y \in \mathbb{R}^L_+$, and $x \sim y$. Then $u(x) = u(y)$ and hence $u(x) + \alpha = u(y) + \alpha$. By the functional form,

$$u(x) + \alpha = (\alpha + x_1) + \phi(x_2,\ldots,x_L) = u(x + \alpha e_1),$$

$$u(y) + \alpha = (\alpha + y_1) + \phi(y_2,\ldots,y_L) = u(y + \alpha e_1),$$

where $e_1 = (1,0,\ldots,0) \in \mathbb{R}^L_+$. Hence $u(x + \alpha e_1) = u(y + \alpha e_1)$, or $x + \alpha e_1 \sim y + \alpha e_1$.

Suppose conversely that $\succsim$ is quasilinear with respect to the first commodity. The idea of the proof of this direction is the same as in (a) or Proposition 3.C.1, in that we reduce comparison of commodity bundles on a line by fining out indifferent bundles and then assigning utility levels along the line. But this proof turns out to exhibit more intricacies, partly because it depends crucially on the connectedness of $\mathbb{R}_+^{L-1}$, which appears in $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$. (Connectedness was mentioned in the first small-type discussion in the proof of Proposition 3.C.1.) The proof will be done in a series of steps. First, we show that comparison of bundles can be reduced to a line parallel to e_1. Then we show that the quasilinearity of $\succsim$ implies the given functional form.

Let $\succsim$ be a quasilinear preference and a utility function $\tilde{u}(\cdot)$ represent $\succsim$. The existence of such a $\tilde{u}(\cdot)$ is guaranteed by Proposition 3.C.1, but, of course, it need not be of the quasilinear form. For each $\hat{x} \in \mathbb{R}_+^{L-1}$, define $I(\hat{x}) = \{u(x_1,\hat{x}) \in \mathbb{R}: x_1 \in \mathbb{R}\}$, then $I(\hat{x})$ is a nonempty open interval, by the continuity and the strong monotonicity of $\succsim$ along e_1.

Step 1: For every $\hat{x} \in \mathbb{R}_+^{L-1}$ *and* $\hat{y} \in \mathbb{R}_+^{L-1}$, *if* $I(\hat{x}) \neq I(\hat{y})$, *then* $I(\hat{x}) \cap I(\hat{y}) = \emptyset$.

Proof: Suppose that $I(\hat{x}) \neq I(\hat{y})$. Without loss of generality, we can assume that there exists $u \in I(\hat{x})$ such that $u \notin I(\hat{y})$. Then either $u \geq \sup I(\hat{y})$ or $u \leq \inf I(\hat{y})$. Suppose that $u \geq \sup I(\hat{y})$. (The other case can be treated similarly.) Then let $x_1^* \in \mathbb{R}$ satisfy $u = \tilde{u}(x_1^*,\hat{x})$, then, for every $y_1 \in \mathbb{R}$, $(x_1^*,\hat{x}) \succ (y_1,\hat{y})$. In particular, for every $x_1 \in \mathbb{R}$ and $y_1 \in \mathbb{R}$, $(x_1^*,\hat{x}) \succ (y_1 - x_1 + x_1^*, \hat{y})$. By the quasilinearity, this implies that $(x_1,\hat{x}) \succ (y_1,\hat{y})$. Thus $\tilde{u}(x_1,\hat{x}) > \tilde{u}(y_1,\hat{y})$. Hence $I(\hat{x}) \cap I(\hat{y}) = \emptyset$.

For each $\hat{x} \in \mathbb{R}_+^{L-1}$, define $E(\hat{x}) = \{\hat{y} \in \mathbb{R}_+^{L-1}: I(\hat{x}) = I(\hat{y})\}$.

Step 2: For every $\hat{x} \in \mathbb{R}_+^{L-1}$, $E(\hat{x})$ *is open in* $\mathbb{R}_+^{L-1}$.

Proof: Let $\hat{x} \in \mathbb{R}_+^{L-1}$, $x_1 \in \mathbb{R}$, and $u = \tilde{u}(x_1,\hat{x}) \in I(\hat{x})$. Let $\varepsilon > 0$ satisfy $(u - \varepsilon, u + \varepsilon) \subset I(\hat{x})$. Since $\tilde{u}(\cdot)$ is continuous, there exists $\delta > 0$ such that if $\hat{y} \in \mathbb{R}_+^{L-1}$ and $\|\hat{x} - \hat{y}\| < \delta$, then $|\tilde{u}(x_1,\hat{x}) - \tilde{u}(x_1,\hat{y})| < \varepsilon$. Hence

$$I(\hat{x}) \cap I(\hat{y}) \supset (u - \varepsilon, u + \varepsilon) \cap I(\hat{y}) \neq \emptyset.$$

Thus, by Step 1, $I(\hat{x}) = I(\hat{y})$, or $\hat{y} \in E(\hat{x})$. Hence $\{\hat{y} \in \mathbb{R}_+^{L-1}: \|\hat{x} - \hat{y}\| < \delta\} \subset E(\hat{x})$. Thus $E(\hat{x})$ is open.

Step 3: For every $\hat{x} \in \mathbb{R}_+^{L-1}$, $E(\hat{x}) = \mathbb{R}_+^{L-1}$.

Proof: It is sufficient to show that for every $\hat{x} \in \mathbb{R}_+^{L-1}$ and $\hat{y} \in \mathbb{R}_+^{L-1}$, we have $E(\hat{x}) = E(\hat{y})$. Suppose not, then there exist $\hat{x} \in \mathbb{R}_+^{L-1}$ and $\hat{y} \in \mathbb{R}_+^{L-1}$ such that $E(\hat{x}) \neq E(\hat{y})$, then the complement $\mathbb{R}_+^{L-1}\backslash E(\hat{x})$ is nonempty. By Step 1, $\mathbb{R}_+^{L-1}\backslash E(\hat{x})$ is equal to the union of those $E(\hat{y})$ for which $\hat{y} \in \mathbb{R}_+^{L-1}\backslash E(\hat{x})$. By Step 2, this implies that $\mathbb{R}_+^{L-1}\backslash E(\hat{x})$ is open. Hence we have obtained a partition $\{E(\hat{x}), \mathbb{R}_+^{L-1}\backslash E(\hat{x})\}$ of $\mathbb{R}_+^{L-1}$, both of whose elements are nonempty and open. This contradicts the connectedness of $\mathbb{R}_+^{L-1}$. Hence $E(\hat{x}) = E(\hat{y})$ for every $\hat{x} \in \mathbb{R}_+^{L-1}$ and $\hat{y} \in \mathbb{R}_+^{L-1}$.

By Step 3, $I(\hat{x}) = I(0)$ for every $\hat{x} \in \mathbb{R}_+^{L-1}$. Thus for every $\hat{x} \in \mathbb{R}_+^{L-1}$, there exists a unique $\alpha \in \mathbb{R}$ such that $\alpha e_1 \sim (0,\hat{x})$. Define ϕ: $\mathbb{R}_+^{L-1} \to \mathbb{R}$ by $\phi(\hat{x})e_1 \sim (0,\hat{x})$ for every $\hat{x} \in \mathbb{R}_+^{L-1}$. Define u: $X \to \mathbb{R}$ by $u(x) = x_1 + \phi(x_2,\ldots,x_L)$ for every $x \in X$.

Step 4: The function $u(\cdot)$ *represents* $\succsim$.

Proof: Suppose that $x \in X$, $y \in X$, and $x \succsim y$. By the quasilinearity, this is equivalent to $(x_1 - y_1, x_2,\ldots, x_L) \succsim (0,y_2,\ldots,y_L)$. By the definition of $\phi(\cdot)$, this is equivalent to $(x_1 - y_1, x_2,\ldots, x_L) \succsim \phi(y_2,\ldots,y_L)e_1$. Again by the quasilinearity, this is equivalent to

$$(0,x_2,\ldots, x_L) \succsim (\phi(y_2,\ldots,y_L) + y_1 - x_1)e_1.$$

Again by the definition of $\phi(\cdot)$, this is equivalent to

$$\phi(x_2,\ldots, x_L)e_1 \succsim (\phi(y_2,\ldots,y_L) + y_1 - x_1)e_1.$$

Hence $\phi(x_2,\ldots, x_L) \geq \phi(y_2,\ldots,y_L) + y_1 - x_1$, that is, $u(x) \geq u(y)$.

These properties of $u(\cdot)$ are cardinal, because they are not preserved under some monotone transformation, such as $f(u(x)) = u(x)^3$.

3.C.6 **(a)** For $\rho = 1$, we have $u(x) = \alpha_1 x_1 + \alpha_2 x_2$. Thus the indifference curves are linear.

(b) Since every monotonic transformations of a utility function represents the same preference, we shall consider

$$\tilde{u}(x) = \ln u(x) = (1/\rho)\ln(\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}).$$

By L'Hopital's rule,

$$\begin{aligned} &\lim_{\rho \to 0} \tilde{u}(x) \\ &= \lim_{\rho \to 0} (\alpha_1 x_1^{\rho} \ln x_1 + \alpha_2 x_2^{\rho} \ln x_2)/(\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}) \\ &= (\alpha_1 \ln x_1 + \alpha_2 \ln x_2)/(\alpha_1 + \alpha_2). \end{aligned}$$

Since $\exp((\alpha_1 + \alpha_2)\tilde{u}(x)) = x_1^{\alpha_1} x_2^{\alpha_2}$, we have obtained a Cobb-Douglas utility function.

There is an alternative proof to this proposition: Since both the CES and the Cobb-Douglas utility functions are continuously differentiable and homothetic, it is sufficient to check the convergence of the marginal rate of substitution at every point. The marginal rate of substitution at (x_1, x_2) with respect to the CES utility function is equal to $\alpha_1 x_1^{\rho-1}/\alpha_2 x_2^{\rho-1}$. The marginal rate of substitution at (x_1, x_2) with respect to the Cobb-Douglas utility function is equal to $\alpha_1 x_1/\alpha_2 x_2$. Note that $\alpha_1 x_1^{\rho-1}/\alpha_2 x_2^{\rho-1} \to \alpha_1 x_1/\alpha_2 x_2$ as $\rho \to 1$. (In fact, $\alpha_1 x_1^{\rho-1}/\alpha_2 x_2^{\rho-1}$ is well defined for every ρ and is equal to $\alpha_1 x_1/\alpha_2 x_2$ when $\rho = 1$.) The proof is thus completed.

Strictly speaking, there is a missing point in both proofs: We proved the convergence of preferences on the strictly positive orthant $\{x \in \mathbb{R}^2 : x \gg 0\}$, but we did not prove the convergence on the horizontal and vertical axes. In fact, the convergence on the axes are obtained in such a way that all vectors there tend to be indifferent. To be more specific, compare, for example, $x = (x_1,0)$ and $y = (y_1,0)$ with $x_1 > y_1 > 0$. According to the CES utility function, x is preferred to y, regardless of the values of ρ. But, according to the Cobb-Douglas utility function, x and y are indifferent. Futhermore, the following is true: If x is in the strictly positive orthant and y is on an axis, then x is preferred to y for every ρ sufficiently close to 0. To see this, simply note that if $x = (x_1,0)$ with $x_1 > 0$ and $y \gg 0$, then $\alpha_1 x_1^\rho \to \alpha_1$ and $\alpha_1 y_1^\rho + \alpha_2 y_2^\rho \to \alpha_1 + \alpha_2$. The implication of this fact is that, as $\rho \to 0$, every vector in the strictly positive orthant becomes preferred to all vectors on the axes. That is, unconditional preference towards strictly positive vectors tends to hold, as it is true for the Cobb-Douglas utility function.

(c) Suppose that $x_1 \le x_2$. We want to show that

$$x_1 = \lim_{\rho \to -\infty} (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{1/\rho}.$$

Let $\rho < 0$. Since $x_1 \ge 0$ and $x_2 \ge 0$, we have $\alpha_1 x_1^\rho \le \alpha_1 x_1^\rho + \alpha_2 x_2^\rho$. Thus $\alpha_1^{1/\rho} x_1 \ge (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{1/\rho}$. On the other hand, since $x_1 \le x_2$,

$$\alpha_1 x_1^\rho + \alpha_2 x_2^\rho \le \alpha_1 x_1^\rho + \alpha_2 x_1^\rho = (\alpha_1 + \alpha_2) x_1^\rho.$$

Hence $(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{1/\rho} \ge (\alpha_1 + \alpha_2)^{1/\rho} x_1$. Therefore,

$$\alpha_1^{1/\rho} x_1 \ge (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{1/\rho} \ge (\alpha_1 + \alpha_2)^{1/\rho} x_1.$$

Letting $\rho \to -\infty$, we obtain $\lim_{\rho \to -\infty} (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{1/\rho} = x_1$, because $\lim_{\rho \to -\infty} \alpha_1^{1/\rho} x_1 = \lim_{\rho \to -\infty} (\alpha_1 + \alpha_2)^{1/\rho} x_1 = x_1$.

3.D.1 To check condition (i),

$$x_1(\lambda p, \lambda w) = \alpha(\lambda w)/(\lambda p_1) = \alpha w/p_1 = x_1(p,w),$$

$$x_2(\lambda p, \lambda w) = (1 - \alpha)(\lambda w)/(\lambda p_2) = (1 - \alpha)w/p_2 = x_2(p,w).$$

To check condition (ii),

$$p_1 x_1(p,w) + p_2 x_2(p,w) = p_1 \alpha w/p_1 + p_2(1 - \alpha)w/p_2 = w.$$

Condition (iii) is obvious.

3.D.2 To check condition (i),

$$\begin{aligned} & v(\lambda p, \lambda w) \\ &= \alpha \ln\alpha + (1 - \alpha)\ln(1 - \alpha) + \ln\lambda w - \alpha \ln\lambda p_1 - (1 - \alpha)\ln\lambda p_2 \\ &= \alpha \ln\alpha + (1 - \alpha)\ln(1 - \alpha) + \ln\lambda + \ln w \\ &\qquad - \alpha\ln\lambda - \alpha\ln p_1 - (1 - \alpha)\ln\lambda - (1 - \alpha)\ln p_2 \\ &= \alpha \ln\alpha + (1 - \alpha)\ln(1 - \alpha) + \ln w - \alpha\ln p_1 - (1 - \alpha)\ln p_2 \\ &= v(p,w). \end{aligned}$$

To check condition (ii),

$$\partial v(p,w)/\partial w = 1/w > 0,$$

$$\partial v(p,w)/\partial p_1 = - \alpha/p_1 < 0,$$

$$\partial v(p,w)/\partial p_2 = - (1 - \alpha)/p_2 < 0.$$

Condition (iv) follows the functional form of $v(\cdot)$.

In order to verify (iii), by property (i), it is sufficient to prove that, for any $v \in \mathbb{R}$ and $w > 0$, the set $\{p \in \mathbb{R}^L_{++}: v(p,w) \le v\}$ is convex. Since the logarithmic function is concave, the set

$$\{(p_1,p_2) \in \mathbb{R}^2_{++}: - \alpha\ln p_1 - (1 - \alpha)\ln p_2 \le v\}$$

is convex for every $v \in \mathbb{R}$. Since the other terms,

$$\alpha\ln\alpha + (1 - \alpha)\ln(1 - \alpha) + \ln w,$$

do not depend on p, this implies that the set $\{p \in \mathbb{R}^L_{++}: v(p,w) \le v\}$ is convex.

3.D.3 **(a)** We shall prove that for every $p \in \mathbb{R}^L_{++}$, $w \ge 0$, $\alpha \ge 0$, and $x \in \mathbb{R}^L_{+}$, if

$x = x(p,w)$, then $\alpha x = x(p,\alpha w)$. Note first that $p\cdot(\alpha x) \le \alpha w$, that is, αx is affordable at $(p,\alpha w)$. Let $y \in \mathbb{R}^L_+$ and $p\cdot y \le \alpha w$. Then $p\cdot(\alpha^{-1}y) \le w$. Hence $u(\alpha^{-1}y) \le u(x)$. Thus, by the homogeneity, $u(y) \le u(\alpha x)$. Hence $\alpha x = x(p,\alpha w)$.

By this result,

$$v(p,\alpha w) = u(x(p,\alpha w)) = u(\alpha x(p,w)) = \alpha u(x(p,w)) = \alpha v(p,w).$$

Thus the indirect utility function is homogeneous of degree one in w.

Given the above results, we can write $x(p,w) = wx(p,1) = w\tilde{x}(p)$ and $v(p,w) = wv(p,1) = w\tilde{v}(p)$. Exercise 2.E.4 showed that the wealth expansion path $\{x(p,w)\colon w > 0\}$ is a ray going through $\tilde{x}(p)$. The wealth elasticity of demand $\varepsilon_{\ell w}$ is equal to 1.

(b) We first prove that for every $p \in \mathbb{R}^L_{++}$, $w \ge 0$, and $\alpha \ge 0$, we have $x(p,\alpha w) = \alpha x(p,w)$. In fact, since $v(\cdot,\cdot)$ is homogeneous of degree one in w, $\nabla_p v(p,\alpha w) = \alpha\nabla_p v(p,w)$ and $\nabla_w v(p,\alpha w) = \nabla_w v(p,w)$. Thus, by Roy's identity, $x(p,\alpha w) = \alpha x(p,w)$.

Now let $x \in \mathbb{R}^L_+$, $x' \in \mathbb{R}^L_+$, $u(x) = u(x')$, and $\alpha \ge 0$. Since $u(\cdot)$ is strictly quasiconcave, by the supporting hyperplane theorem (Theorem M.G.3), there exist $p \in \mathbb{R}^L_{++}$, $p' \in \mathbb{R}^L_{++}$, $w \ge 0$, and $w' \ge 0$ such that $x = x(p,w)$ and $x' = x(p',w')$. Then $u(x) = v(p,w)$ and $u(x') = v(p',w')$. Hence $v(p,w) = v(p',w')$. Thus, by the homogeneity, $v(p,\alpha w) = v(p',\alpha w')$. But as we saw above, $x(p,\alpha w) = \alpha x$ and $x(p',\alpha w') = \alpha x'$. Hence $v(p,\alpha w) = u(\alpha x)$ and $v(p',\alpha w') = u(\alpha x')$. Thus $u(\alpha x) = u(\alpha x')$. Therefore $u(x)$ is homogeneous of degree one.

3.D.4 **(a)** Let $e_1 = (1,0,\ldots,0) \in \mathbb{R}^L$. We shall prove that for every $p \in \mathbb{R}^L_{++}$, $w \in \mathbb{R}$, $\alpha \in \mathbb{R}$, and $x \in (-\infty, \infty) \times \mathbb{R}^{L-1}_+$, if $x = x(p,w)$, then $x + \alpha e_1 = x(p, w + \alpha)$. Note first that, by $p_1 = 1$, $p\cdot(\alpha x + e_1) \le \alpha w$, that is, $x + \alpha e_1$ is affordable at $(p, w + \alpha e_1)$. Let $y \in \mathbb{R}^L_+$ and $p\cdot y \le w + \alpha$. Then $p\cdot(y - \alpha e_1)$

$\leq$ w. Hence $x \succsim y - \alpha e_1$. Thus, by the quasilinearity, $x + \alpha e_1 \succsim y$. Hence $x + \alpha e_1 = x(p, w + \alpha)$.

Therefore, for every $\ell \in \{2,\dots,L\}$, $w \in \mathbb{R}$, and $w' \in \mathbb{R}$, $x_\ell(p,w) = x_\ell(p,w')$. That is, the Walrasian demand functions for goods 2,...,L are independent of wealth. As for good 1, we have $\partial x(p,w)/\partial w = 1$ for every (p,w). That is, any additional amount of money is spent on good 1.

(b) Define $\phi(p) = u(x(p,0))$. Since $x(p,w) = x(p,0) + we_1$ and the preference relation can be represented by a utility function of the quasilinear form $u(x) = x_1 + \tilde{u}(x_2,\dots,x_L)$ (Exercise 3.C.5), we have

$$\begin{aligned} v(p,w) &= u(x(p,w)) \\ &= x_1(p,w) + \tilde{u}(x_2(p,w),\dots,x_L(p,w)). \\ &= w + x_1(p,0) + \tilde{u}(x_2(p,0),\dots,x_L(p,0)) \\ &= w + u(x(p,0)) = w + \phi(p). \end{aligned}$$

(c) The non-negativity constraint is binding if and only if $p_2 x_2(p,0) > w$. Note that $x_2(p,0) = (\eta')^{-1}(p_2)$, because $p_1 = 1$. Hence the constraint is binding if and only if $p_2(\eta')^{-1}(p_2) > w$. If so, the Walrasian demand is given by $x(p,w) = (0, w/p_2)$. Thus, as w changes, the consumption level of the first good is unchanged and the consumption of the second good changes at rate $1/p_2$ with w until the non-negativity constraint no longer binds.

3.D.5 (a) Since any monotone transformation of a utility function represents the same preference relation, we may as well choose

$$\tilde{u}(x) = \rho u(x)^\rho = \rho(x_1^\rho + x_2^\rho).$$

By the first-order condition of the UMP with $\tilde{u}(\cdot)$,

$$x(p,w) = (w/(p_1^\delta + p_2^\delta))(p_1^{\delta-1}, p_2^{\delta-1}),$$

where $\delta = \rho/(\rho - 1) \in (-\infty, 1)$. Plug this into $u(\cdot)$, then we obtain

$$v(p,w) = w/(p_1^\delta + p_2^\delta)^{1/\delta}.$$

(b) To check the homogeneity of the demand function,

$$x(\alpha p,\alpha w) = (\alpha w/((\alpha p_1)^\delta + (\alpha p_2)^\delta)((\alpha p_1)^{\delta-1},(\alpha p_2)^{\delta-1})$$
$$= (\alpha\cdot\alpha^{\delta-1}/\alpha^\delta)(w/(p_1^\delta + p_2^\delta))(p_1^{\delta-1},p_2^{\delta-1})$$
$$= x(p,w).$$

To check Walras' law,

$$p\cdot x(p,w) = (w/(p_1^\delta + p_2^\delta))(p_1\cdot p_1^{\delta-1} + p_2\cdot p_2^{\delta-1}) = w.$$

The uniqueness is obvious.

To check the homogeneity of the indirect utility function,

$$v(\alpha p,\alpha w)$$
$$= \alpha w/((\alpha p_1)^\delta + (\alpha p_2)^\delta)^{1/\delta} = \alpha w/\alpha^{\delta\cdot 1/\delta}(p_1^\delta + p_2^\delta)^{1/\delta} = w/(p_1^\delta + p_2^\delta)^{1/\delta}$$
$$= v(p,w)$$

To check the monotonicity,

$$\partial v(p,w)/\partial w = 1/(p_1^\delta + p_2^\delta)^{1/\delta} > 0,$$
$$\partial v(p,w)/\partial p_\ell = - wp_\ell^{\delta-1}/(p_1^\delta + p_2^\delta)^{1/\delta+1} < 0.$$

The continuity follows immediately from the derived functional form.

In order to prove the quasiconvexity, by property the homogeneity, it is sufficient to prove that, for any $v \in \mathbb{R}$ and $w > 0$, the set $\{p \in \mathbb{R}^2: v(p,w) \le v\}$ is convex. If $\delta = 0$, then the utility function is a Cobb-Douglas one, and the quasiconcavity was already established in Exercise 3.D.2. So we consider two cases, $\delta \in (0,1)$ and $\delta < 0$. In either case, define $f(p) = (p_1^\delta + p_2^\delta)^{1/\delta}$.

If $\delta \in (0,1)$, then $f(p)^\delta = p_1^\delta + p_2^\delta$ is a concave function. Hence $\{p \in \mathbb{R}^2: f(p) = (f(p)^\delta)^{1/\delta} \ge v\}$ is convex for every v. Since $v(p,w) = w/f(p)$, this implies that $\{p \in \mathbb{R}^2: v(p,w) \le v\}$ is convex for every v and w.

If $\delta < 0$, then $f(p)^\delta = p_1^\delta + p_2^\delta$ is a convex function. Hence $\{p \in \mathbb{R}^2: 1/f(p) = (f(p)^\delta)^{1/(-\delta)} \le v\}$ is convex for every v. Since $v(p,w) = w/f(p)$, this implies that $\{p \in \mathbb{R}^2: v(p,w) \le v\}$ is convex for every v and w.

(c) For the linear indifference curves, we have

$$x(p,w) = \begin{cases} (w/p_1,\ 0) & \text{if } p_1 < p_2, \\ (0,\ w/p_2) & \text{if } p_1 > p_2, \\ \{(w/p_1)(\lambda,\ 1-\lambda): \lambda \in [0,1]\} & \text{if } p_1 = p_2; \end{cases}$$

$$v(p,w) = \max\{w/p_1, w/p_2\}.$$

For the Leontief preference,

$$x_1(p,w) = (w/(p_1 + p_2))(1,1);$$

$$v(p,w) = w/(p_1 + p_2).$$

As for the limit argument with respect to ρ. First consider the case with $\rho < 1$ and $\rho \to 1$. Then $\delta = \rho/(\rho - 1) \to -\infty$ as $\rho \to 1$.

Case 1. $p_1 < p_2$.

Since $p_2/p_1 > 1$, we have $(p_2/p_1)^\delta \to 0$. Thus

$$\lim_{\delta \to -\infty} p_1^{\delta-1} w/(p_1^\delta + p_2^\delta) = \lim_{\delta \to -\infty} \frac{w/p_1}{1 + (p_2/p_1)^\delta} = w/p_1.$$

Since $p_1/p_2 < 1$, we have $(p_1/p_2)^\delta \to \infty$. Thus

$$\lim_{\delta \to -\infty} p_2^{\delta-1} w/(p_1^\delta + p_2^\delta) = \lim_{\delta \to -\infty} \frac{w/p_2}{(p_1/p_2)^\delta + 1} = 0.$$

Thus the CES Walrasian demands converge to the Walrasian demand of the linear indifference curves. As for the indirect utility functions, we showed in the answer to Exercise 3.C.6(c) that $(p_1^\delta + p_2^\delta)^{1/\delta} \to p_1$ for $p_1 \le p_2$. Hence the CES indirect utilities converge to the indirect utility of the linear indifference curves.

Case 2. $p_1 > p_2$.

Do the same argument as in the Case 1.

Case 3. $p_1 = p_2$.

In this case, $(w/(p_1^\delta + p_2^\delta))(p_1^{\delta-1}, p_2^{\delta-1}) = (w/(p_1^\delta + p_1^\delta))(p_1^{\delta-1}, p_1^{\delta-1}) = (w/2p_1)(1,1)$. This consumption bundle belongs to the set of the Walrasian demands of the linear indifference curves when $p_1 = p_2$. As for the indirect utility functions, we showed in the answer to Exercise 3.C.6(c) that $(p_1^\delta + p_2^\delta)^{1/\delta} \to p_1$ for $p_1 \le p_2$.

Let's next consider the case $\rho \to -\infty$. Note that $\delta = \rho/(\rho - 1) \to 1$ as $\rho \to 1$. So just plug $\delta = 1$ into the CES Walrasian demand functions and the indirect utility functions. We then get the Walrasian demand function and the indirect utility function of the Leontief preference.

(d) From the calculation of the Walrasian demand functions in (a) we get

$$x_1(p,w)/x_2(p,w) = (p_1/p_2)^{\delta-1},$$
$$(x_1(p,w)/x_2(p,w))/(p_1/p_2) = (p_1/p_2)^{\delta-2},$$
$$d[x_1(p,w)/x_2(p,w)]/d[p_1/p_2] = (\delta - 1)(p_1/p_2)^{\delta-2}.$$

Thus $\xi_{12}(p,w) = -(\delta - 1) = 1/(1 - \rho)$. Hence, $\xi_{12}(p,w) = \infty$ for the linear, $\xi_{12}(p,w) = 0$ for the Leontief, and $\xi_{12}(p,w) = 1$ for the Cobb-Douglas utility functions.

3.D.6 **(a)** Define $\tilde{u}(x) = u(x)^{1/(\alpha+\beta+\gamma)} = (x_1 - b_1)^{\alpha'}(x_2 - b_2)^{\beta'}(x_3 - b_3)^{\gamma'}$, with $\alpha' = \alpha/(\alpha + \beta + \gamma)$, $\beta' = \beta/(\alpha + \beta + \gamma)$, $\gamma' = \gamma/(\alpha + \beta + \gamma)$. Then $\alpha' + \beta' + \gamma' = 1$ and $\tilde{u}(\cdot)$ represents the same preferences as $u(\cdot)$, because the function $u \to u^{1/(\alpha+\beta+\gamma)}$ is a monotone transformation. Thus we can assume without loss of generality that $\alpha + \beta + \gamma = 1$.

(b) Use another monotone transformation of the given utility function,

$$\ln u(x) = \alpha\ln(x_1 - b_1) + \beta\ln(x_2 - b_2) + \gamma\ln(x_3 - b_3).$$

The first-order condition of the UMP yields the demand function

$$x(p,w) = (b_1,b_2,b_3) + (w - p\cdot b)(\alpha/p_1,\beta/p_2,\gamma/p_3),$$

where $p\cdot b = p_1b_1 + p_2b_2 + p_3b_3$. Plug this demand function to $u(\cdot)$, then we obtain the indirect utility function

$$v(p,w) = (w - p\cdot b)(\alpha/p_1)^{\alpha}(\beta/p_2)^{\beta}(\gamma/p_3)^{\gamma}.$$

(c) To check the homogeneity of the demand function,

$$x(\lambda p,\lambda w) = (b_1,b_2,b_3) + (\lambda w - \lambda p\cdot b)(\alpha/\lambda p_1,\beta/\lambda p_2,\gamma/\lambda p_3)$$
$$= (b_1,b_2,b_3) + (w - p\cdot b)(\alpha/p_1,\beta/p_2,\gamma/p_3) = x(p,w).$$

To check Walras law,

$$p\cdot x(p,w) = p\cdot b + (w - p\cdot b)(p_1\alpha/p_1 + p_2\beta/p_2 + p_3\gamma/p_3)$$
$$= p\cdot b + (w - p\cdot b)(\alpha + \beta + \gamma) = w.$$

The uniqueness is obvious.

To check the homogeneity of the indirect utility function,

$$v(\lambda p,\lambda w) = (\lambda w - \lambda p\cdot b)(\alpha/\lambda p_1)^{\alpha}(\beta/\lambda p_2)^{\beta}(\gamma/\lambda p_3)^{\gamma}$$
$$= \lambda^{1-(\alpha+\beta+\gamma)}(w - p\cdot b)(\alpha/p_1)^{\alpha}(\beta/p_2)^{\beta}(\gamma/p_3)^{\gamma}$$
$$= (w - p\cdot b)(\alpha/p_1)^{\alpha}(\beta/p_2)^{\beta}(\gamma/p_3)^{\gamma} = v(p,w).$$

To check the monotonicity,

$$\partial v(p,w)/\partial w = (\alpha/p_1)^{\alpha}(\beta/p_2)^{\beta}(\gamma/p_3)^{\gamma} > 0,$$
$$\partial v(p,w)/\partial p_1 = v(p,w)\cdot(-\ \alpha/p_1) < 0,$$
$$\partial v(p,w)/\partial p_2 = v(p,w)\cdot(-\ \beta/p_2) < 0,$$
$$\partial v(p,w)/\partial p_3 = v(p,w)\cdot(-\ \gamma/p_3) < 0.$$

The continuity follows directly from the given functional form. In order to prove the quasiconvexity, it is sufficient to prove that, for any $v \in \mathbb{R}$ and $w > 0$, the set $\{p \in \mathbb{R}^3\colon v(p,w) \le v\}$ is convex. Consider

$$\ln v(p,w) = \alpha\ln\alpha + \beta\ln\beta + \gamma\ln\gamma + \ln(w - p\cdot b) - \alpha\ln p_1 - \beta\ln p_2 - \gamma\ln p_3.$$

Since the logarithmic function is concave, the set

$$\{p \in \mathbb{R}^3: \ln(w - p\cdot b) - \alpha\ln p_1 - \beta\ln p_2 - \gamma\ln p_3 \le v\}$$

is convex for every $v \in \mathbb{R}$. Since the other terms, $\alpha\ln\alpha + \beta\ln\beta + \gamma\ln\gamma$, do not depend on p, this implies that the set $\{p \in \mathbb{R}^3: \ln v(p,w) \le v\}$ is convex. Hence so is $\{p \in \mathbb{R}^3: v(p,w) \le v\}$

3.D.7 **(a)** Since $p^1\cdot x^0 < w^1$ and $x^1 \ne x^0$, the weak axiom implies $p^0\cdot x^1 > w^0$. Thus x^1 has to be on the bold line in the following figure.

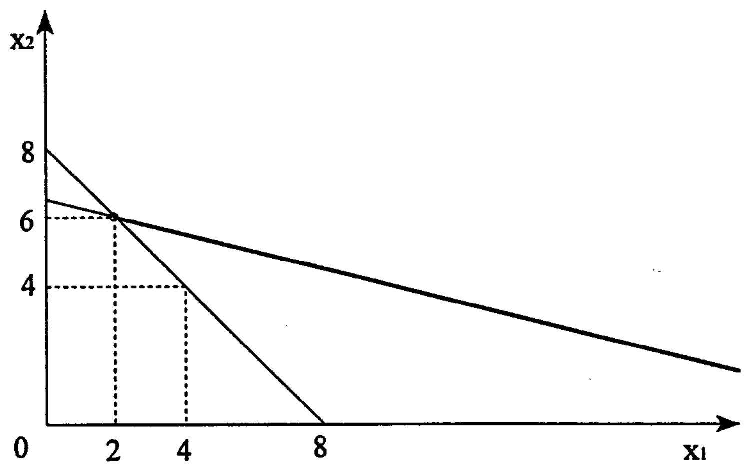


Figure 3.D.7(a)

In the following four question, we assume the given preference can be a differentiable utility function $u(\cdot)$.

(b) If the preference is quasilinear with respect to the first good, then we can take a utility function $u(\cdot)$ so that $\partial u(x)/\partial x_1 = 1$ for every x (Exercise 3.C.5(b)). Hence the first-order condition implies $\partial u(x^t)/\partial x_2^t = p_2^t/p_1^t$ for each $t = 0,1$. Since $p_2^0/p_1^0 < p_2^1/p_1^1$ and $u(\cdot)$ is concave, $x_2^0 > x_2^1$. Thus x^1 has to be on the bold line in the following figure.

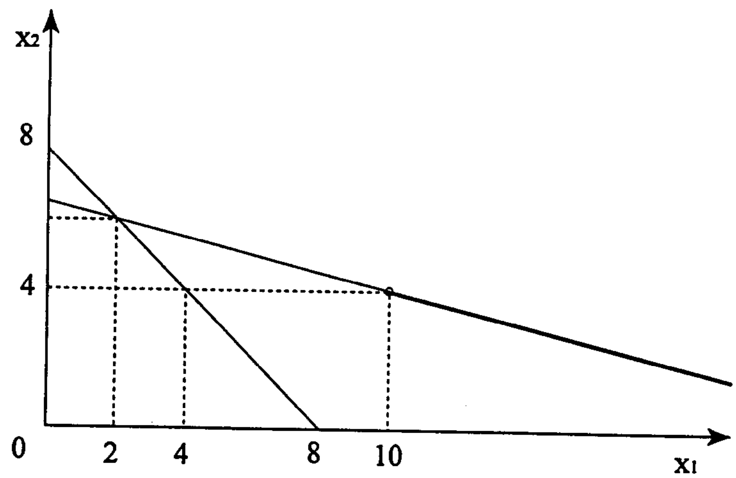


Figure 3.D.7(b)

(c) If the preference is quasilinear with respect to the second good, then then we can take a utility function $u(\cdot)$ so that $\partial u(x)/\partial x_2 = 1$ for every x (Exercise 3.C.5(b)). Hence the first-order condition implies $\partial u(x^t)/\partial x_1^t = p_1^t/p_2^t$ for each t = 0,1. Since $p_1^0/p_2^0 > p_1^1/p_2^1$ and $u(\cdot)$ is concave, we must have $x_1^0 < x_1^1$. Thus x^1 has to be on the bold line in the following figure.

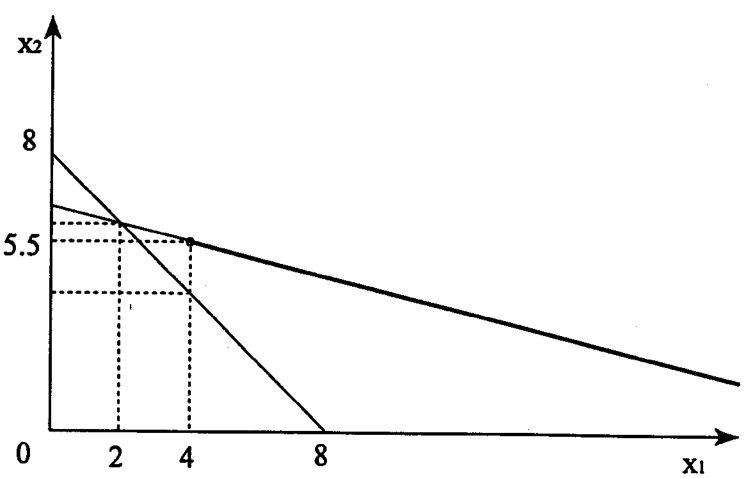


Figure 3.D.7(c)

(d) Since $p^1 \cdot x^0 < w^1$ and the relative price of good 1 decreased, x_1^t has to increase if good 1 is normal. If good 2 is normal, then the wealth effect (positive) and the substitution effect (negative) go in opposite direction which gives us no additional information about x_2. Thus x^1 has to be on the bold line in the following figure.

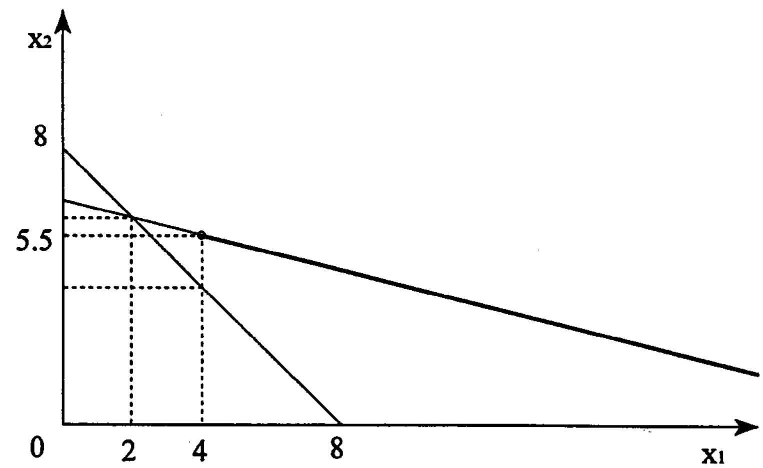


Figure 3.D.7(d)

(e) If the preference is homothetic, the the marginal rates of substitution at all vectors on a ray are the same, and they becomes less steep as the ray becomes flatter. By the first-order conditions and $p_1^0/p_2^0 > p_1^1/p_2^1$, x^1 has to be on the right side of the ray that goes through x^0. Thus x^1 has to be on the bold line in the following figure.

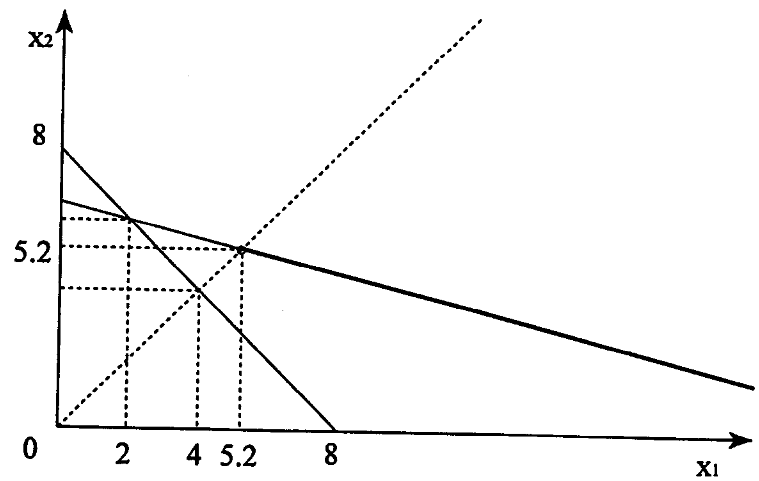


Figure 3.D.7(e)

3.D.8 By Proposition 3.D.3(i), $v(\alpha p, \alpha w) = v(p,w)$ for all $\alpha > 0$. By differentiating this equality with respect to α and evaluating at $\alpha = 1$, we obtain $\nabla_p v(p,w)\cdot p + w\partial v(p,w)/\partial w = 0$. Thus $w\partial v(p,w)/\partial w = -\nabla_p v(p,w)\cdot p$.

3.E.1 The EMP is equivalent to the following maximization problem:

$$\text{Max} \quad -p\cdot x \quad \text{s.t.} \quad u(x) \geq u \text{ and } x \geq 0.$$

The Kuhn-Tucker condition (Theorem M.K.2) implies that the first-order conditions are that there exists $\lambda > 0$ and $\mu \in \mathbb{R}^L_+$ such that $p = \lambda\nabla u(x^*) + \mu$ and $\mu\cdot x^* = 0$. That is, for some $\lambda > 0$, $p \leq \lambda\nabla u(x^*)$ and $x^*\cdot(p - \lambda\nabla u(x^*)) = 0$. This is the same as that of the UMP.

3.E.2 To check the homogeneity of the expenditure function,

$$e(\lambda p,u) = \alpha^{-\alpha}(1-\alpha)^{\alpha-1}(\lambda p_1)^{\alpha}(\lambda p_2)^{1-\alpha}u$$
$$= \alpha^{-\alpha}(1-\alpha)^{\alpha-1}\lambda^{\alpha+1-\alpha}p_1^{\alpha}p_2^{1-\alpha}u = e(p,u).$$

To check the monotonicity,

$$\partial e(p,u)/\partial u = \alpha^{-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha} > 0,$$

$$\partial e(p,u)/\partial p_1 = \alpha^{1-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha-1}p_2^{1-\alpha} > 0,$$

$$\partial e(p,u)/\partial p_2 = \alpha^{\alpha}(1-\alpha)^{\alpha}p_1^{\alpha}p_2^{-\alpha} > 0.$$

To check the concavity, it is easy to actually calculate $D_p^2 e(p,u)$ and then apply the condition in Exercise 2.F.9 to show that $D_p^2 e(p,u)$ is negative semidefinite. An alternative way is to only calculate

$$\partial^2 e(p,u)/\partial p_1^2 = -\alpha^{1-\alpha}(1-\alpha)^{\alpha}p_1^{\alpha-2}p_2^{1-\alpha} < 0.$$

Then note that the homogeneity implies that $D_p^2 e(p,u)p = 0$. Hence we can apply Theorem M.D.4(iii) to conclude that $D_p^2 e(p,u)$ is negative semidefinite. The continuity follows from the functional form.

To check the homogeneity of the Hicksian demand function,

$$h_1(\lambda p,u) = \left(\frac{\alpha\lambda p_2}{(1-\alpha)\lambda p_1}\right)^{1-\alpha} u = \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha} u = h_1(p,u),$$

$$h_2(\lambda p,u) = \left(\frac{(1-\alpha)\lambda p_1}{\alpha\lambda p_2}\right)^{\alpha} u = \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha} u = h_2(p,u).$$

To check the no excess utility,

$$u(h(p,u)) = \left(\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha} u\right)^{\alpha}\left(\left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha} u\right)^{1-\alpha}$$

$$= \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{(1-\alpha)\alpha-\alpha(1-\alpha)} u^{\alpha+(1-\alpha)} = u.$$

The uniqueness is obvious.

3.E.3 Let $\bar{x} \in \mathbb{R}_+^L$ and $u(\bar{x}) \ge u$. Define $A = \{x \in \mathbb{R}_+^L: p\cdot x \le p\cdot\bar{x} \text{ and } u(x) \ge u\}$. Then $A \ne \varnothing$ by $\bar{x} \in A$. Furthermore, A is compact: The closedness follows from that of $\{x \in \mathbb{R}^L: u(x) \ge u\}$ and of $\mathbb{R}_+^L$; the boundedness follows from the inclusion

$$A \subset \{x \in \mathbb{R}^L: 0 \le x_\ell \le p\cdot\bar{x}/p_\ell \text{ for every } \ell = 1,\ldots,L\}.$$

Now consider the truncated EMP:

$$\text{Min} \quad p \cdot x \quad \text{s.t.} \quad x \in A.$$

Since $p \cdot x$ is a continuous function and A is a compact set, this problem has a solution, denoted by $x^* \in A$. We shall show that this is also a solution to the original EMP. Let $x \in \mathbb{R}^L_+$ and $u(x) \geq u$. If $x \in A$, then $p \cdot x \geq p \cdot x^*$ because x^* is a solution to the truncated EMP. If $x \notin A$, then $p \cdot x > p \cdot \bar{x}$ and hence $p \cdot x > p \cdot x^*$. Thus x^* is a solution of the original EMP.

3.E.4 Suppose first that $\succsim$ is convex and that $x \in h(p,u)$ and $x' \in h(p,u)$. Then $p \cdot x = p \cdot x'$ and $u(x) \geq u$, $u(x') \geq u$. Let $\alpha \in [0,1]$ and define $x'' = \alpha x + (1 - \alpha)x'$. Then $p \cdot x'' = \alpha p \cdot x + (1 - \alpha)p \cdot x' = p \cdot x = p \cdot x'$ and, by the convexity of $\succsim$, $u(x'') \geq u$. Thus $x'' \in h(p,u)$.

Suppose next that $\succsim$ is strictly convex and that $x \in h(p,u)$, $x' \in h(p,u)$, $x \neq x'$, and $u(x) \geq u(x') \geq u$. By the argument above, $x'' = \alpha x + (1 - \alpha)x'$ with $\alpha \in (0,1)$ satisfies $p \cdot x'' = p \cdot x = p \cdot x'$ and, by the strict convexity of $\succsim$, we have $x'' \succ x'$. Since $\succsim$ is continuous, $\beta x'' \succ x'$ for any $\beta \in (0,1)$ close enough to 1. But this implies that $p \cdot (\beta x'') < p \cdot x$ and $u(\beta x'') > u(x') \geq u$, which contradicts the fact that x is a solution of the EMP. Hence $h(p,u)$ must be a singleton.

3.E.5 [<u>First printing errata</u>: The equality $h(p,u) = \tilde{h}(p)u$ should be $h(p,u) = u\tilde{h}(p)$, because u is a scalar and $\tilde{h}(p)$ is a vector.] We shall first prove that, for every $p \gg 0$, $u \geq 0$, $\alpha \geq 0$, and $x \geq 0$, if $x = h(p,u)$, then $\alpha x = h(p,\alpha u)$. In fact, note that $u(\alpha x) = \alpha u(x) \geq \alpha u$, that is, αx satisfies the constraint of the EMP for αu. Let $y \in \mathbb{R}^L_+$ and $u(y) \geq \alpha u$. Then $u(\alpha^{-1}y) \geq u$. Hence $p \cdot (\alpha^{-1}y) \geq p \cdot x$. Thus $p \cdot y \geq p \cdot (\alpha x)$. Hence $\alpha x = h(p,\alpha u)$. Therefore

$h(p,u)$ is homogeneous of degree one in u.

By this result,

$$e(p,\alpha u) = p \cdot h(p,\alpha u) = p \cdot (\alpha h(p,u)) = \alpha(p \cdot h(p,u)) = \alpha e(p,u).$$

Thus the expenditure function is homogeneous of degree one in u.

Now define $\tilde{h}(p) = h(p,1)$ and $\tilde{e}(p) = e(p,1)$, then $h(p,u) = u\tilde{h}(p)$ and $e(p,u) = u\tilde{e}(p)$.

3.E.6 Define $\delta = \rho/(\rho - 1)$, then the expenditure function and the Hicksian demand function are derived from the first-order conditions of the EMP and they are as follows:

$$h(p,u) = u(p_1^\delta + p_2^\delta)^{(1-\delta)/\delta}(p_1^{\delta-1}, p_2^{\delta-1}),$$
$$e(p,u) = u(p_1^\delta + p_2^\delta)^{1/\delta}.$$

To check the homogeneity of the expenditure function,

$$e(\alpha p,u) = u((\alpha p_1)^\delta + (\alpha p_2)^\delta)^{1/\delta} = \alpha^{\delta \cdot 1/\delta} u(p_1^\delta + p_2^\delta)^{1/\delta} = \alpha e(p,u).$$

To check the monotonicity,

$$\partial e(p,u)/\partial u = (p_1^\delta + p_2^\delta)^{1/\delta} > 0,$$
$$\partial e(p,u)/\partial p_\ell = u p_\ell^{\delta-1}(p_1^\delta + p_2^\delta)^{1/\delta-1} > 0.$$

To check the concavity, it is a bit lengthy but easy to actually calculate $D_p^2 e(p,u)$ and then apply the condition in Exercise 2.F.9 to show that $D_p^2 e(p,u)$ is negative semidefinite. An alternative way is to only calculate

$$\partial^2 e(p,u)/\partial p_1^2$$
$$= u(\delta - 1)p_1^{\delta-2}(p_1^\delta + p_2^\delta)^{1/\delta-1} + u p_1^{\delta-1}(p_1^\delta + p_2^\delta)^{1/\delta-2}(1/\delta - 1)\delta p_1^{\delta-1}$$
$$= u(1 - \delta)p_1^{\delta-2}(p_1^\delta + p_2^\delta)^{1/\delta-2}(p_1^\delta - (p_1^\delta + p_2^\delta)) < 0,$$

by $\delta < 1$. Then note that the homogeneity implies that $D_p^2 e(p,u)p = 0$. Hence we can apply Theorem M.D.4(iii) to conclude that $D_p^2 e(p,u)$ is negative semidefinite. The continuity follows from the functional form.

To check the homogeneity of the Hicksian demand function,

$$h(\alpha p,u) = u((\alpha p_1)^{\delta} + (\alpha p_2)^{\delta})^{(1-\delta)/\delta}((\alpha p_1)^{\delta-1},(\alpha p_2)^{\delta-1})$$

$$= \alpha^{(\delta(1-\delta)/\delta)+(\delta-1)}u(p_1^{\delta} + p_2^{\delta})^{(1-\delta)/\delta}(p_1^{\delta-1},p_2^{\delta-1})$$

$$= h(p,u).$$

To check no excess utility,

$$u(h(p,u)) = u(p_1^{\delta} + p_2^{\delta})^{(1-\delta)/\delta}(p_1^{(\delta-1)\rho} + p_2^{(\delta-1)\rho})^{1/\rho}.$$

Since $(\delta - 1)/\delta = -1/\rho$, we obtain $u(h(p,u)) = u$. The uniqueness is obvious.

3.E.7 In Exercise 3.C.5(b), we showed that every quasilinear preference with respect to good 1 can be represented by a utility function of the form $u(x) = x_1 + \tilde{u}(x_2,\ldots,x_L)$. Let $e_1 = (1,0,\ldots,0) \in \mathbb{R}^L$. We shall prove that for every $p \gg 0$ with $p_1 = 1$, $u \in \mathbb{R}$, $\alpha \in \mathbb{R}$, and $x \in (-\infty, \infty) \times \mathbb{R}_+^{L-1}$, if $x = h(p,u)$, then $x + \alpha e_1 = h(p, u + \alpha)$. Note first that $u(x + \alpha e_1) \geq u + \alpha$, that is, $x + \alpha e_1$ satisfies the constraint of the EMP for $(p, u + \alpha)$. Let $y \in \mathbb{R}_+^L$ and $u(y) \geq u + \alpha$. Then $u(y - \alpha e_1) \geq u$. Hence $p\cdot(y - \alpha e_1) \geq p\cdot x$. Thus $p\cdot y \geq p\cdot(x + \alpha e_1)$. Hence $x + \alpha e_1 = h(p, u + \alpha)$.

Therefore, for every $\ell \in \{2,\ldots,L\}$, $u \in \mathbb{R}$, and $u' \in \mathbb{R}$, $h_\ell(p,u) = h_\ell(p,u')$. That is, the Hicksian demand functions for goods $2,\ldots,L$ are independent of utility levels. Thus, if we define $\tilde{h}(p) = h(p,0)$, then $h(p,u) = \tilde{h}(p) + ue_1$.

Since $h(p, u + \alpha) = h(p,u) + \alpha e_1$, we have $e(p, u + \alpha) = e(p,u) + \alpha$. Thus, if we define $\tilde{e}(p) = e(p,0)$, then $e(p,u) = \tilde{e}(p) + u$.

3.E.8 We use the utility function $u(x) = x_1^{\alpha}x_2^{(1-\alpha)}$. To prove (3.E.1),

$$e(p,v(p,w)) = \alpha^{-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}(\alpha^{\alpha}(1-\alpha)^{1-\alpha}p_1^{-\alpha}p_2^{\alpha-1}w) = w,$$

$$v(p,e(p,u)) = \alpha^{\alpha}(1-\alpha)^{1-\alpha}p_1^{-\alpha}p_2^{\alpha-1}(\alpha^{-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}u) = u.$$

To prove (3.E.3),

$$x(p,e(p,u)) = (\alpha^{-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}u)(\alpha/p_1,\ (1-\alpha)/p_2)$$

$$= \left(\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha}u,\ \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha}u\right) = h(p,u),$$

$$h(p,v(p,w)) = \alpha^{\alpha}(1-\alpha)^{1-\alpha}p_1^{-\alpha}p_2^{\alpha-1}w\left(\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha},\ \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha}\right)$$

$$= w(\alpha/p_1,\ (1-\alpha)/p_2) = x(p,w).$$

3.E.9 First, we shall prove that Proposition 3.D.3 implies Proposition 3.E.2 via (3.E.1). Let $p \gg 0$, $p' \gg 0$, $u \in \mathbb{R}$, $u' \in \mathbb{R}$, and $\alpha \geq 0$.

(i) Homogeneity of degree one in p: Let $\alpha > 0$. Define $w = e(p,u)$, then $u = v(p,w)$ by the second relation of (3.E.1). Hence

$$e(\alpha p,u) = e(\alpha p,v(p,w)) = e(\alpha p,v(\alpha p,\alpha w)) = \alpha w = \alpha e(p,u),$$

where the second equality follows from the homogeneity of $v(\cdot,\cdot)$ and the third from the first relation of (3.E.1).

(ii) Monotonicity: Let $u' > u$. Define $w = e(p,u)$ and $w' = e(p,u')$, then $u = v(p,w)$ and $u' = v(p,w')$. By the monotonicity of $v(\cdot,\cdot)$ in w, we must have $w' > w$, that is, $e(p',u) > e(p,u)$.

Next let $p' \geq p$. Define $w = e(p,u)$ and $w' = e(p',u)$, then, by the second relation of (3.E.1), $u = v(p,w) = v(p',w')$. By the monotonicity of $v(\cdot,\cdot)$, we must have $w' \geq w$, that is, $e(p',u) \geq e(p,u)$.

(iii) Concavity: Let $\alpha \in [0,1]$. Define $w = e(p,u)$ and $w' = e(p',u)$, then $u = v(p,w) = v(p',w)$. Define $p'' = \alpha p + (1-\alpha)p''$ and $w'' = \alpha w + (1-\alpha)w'$. Then, by the quasiconvexity of $v(\cdot,\cdot)$, $v(p'',w'') \leq u$. Hence, by the monotonicity of $v(\cdot,\cdot)$ in w and the second relation of (3.E.1), $w'' \leq e(p'',u)$. that is,

$$e(\alpha p + (1-\alpha)p',\ u) \geq \alpha e(p,u) + (1-\alpha)e(p',u).$$

(iv) Continuity: It is sufficient to prove the following statement: For any sequence $\{(p^n,u^n)\}_{n=1}^{\infty}$ with $(p^n,u^n) \to (p,u)$ and any w, if $e(p^n,u^n) \leq w$ for every n, then $e(p,u) \leq w$; if $e(p^n,u^n) \geq w$ for every n, then $e(p,u) \geq w$. Suppose that $e(p^n,u^n) \leq w$ for every n. Then, by the monotonicity of $v(\cdot,\cdot)$ in w, and the second relation of (3.E.1), we have $u^n \leq v(p^n,w)$ for every n. By the continuity of $v(\cdot,\cdot)$, $u \leq v(p,w)$. By the second relation of (3.E.1) and the monotonicity of $v(\cdot,\cdot)$ in w, we must have $e(p,u) \leq w$. The same argument can be applied for the case with $e(p^n,u^n) \geq w$ for every n.

Let's next prove that Proposition 3.E.2 implies Proposition 3.D.3 via (3.E.1). Let $p \gg 0$, $p' \gg 0$, $w \in \mathbb{R}$, $w' \in \mathbb{R}$, and $\alpha \geq 0$.

(i) Homogeneity: Let $\alpha > 0$. Define $u = v(p,w)$. Then, by the first relation of (3.E.1), $e(p,u) = w$. Hence

$$v(\alpha p,\alpha w) = v(\alpha p,\alpha e(p,w)) = v(\alpha p,e(\alpha p,u)) = u = v(p,w),$$

where the second equality follows from the homogeneity of $e(\cdot,\cdot)$ and the third from the second relation of (3.E.1).

(ii) Monotonicity: Let $w' > w$. Define $u = v(p,w)$ and $u' = v(p,w')$, then $e(p,u) = w$ and $e(p,u') = w'$. By the monotonicity of $e(\cdot,\cdot)$ and $w' > w$, we must have $u' > u$, that is, $v(p,w') > v(p,w)$.

Next, assume that $p' \geq p$. Define $u = v(p,w)$ and $u' = v(p',w)$, then $e(p,u) = e(p',u') = w$. By the monotonicity of $e(\cdot,\cdot)$ and $p' \geq p$, we must have $u' \leq u$, that is, $v(p,w) \geq v(p',w)$.

(iii) Quasiconvexity: Let $\alpha \in [0,1]$. Define $u = v(p,w)$ and $u' = v(p',w')$. Then $e(p,u) = w$ and $e(p,u') = w'$. Without loss of generality, assume that $u' \geq u$. Define $p'' = \alpha p + (1 - \alpha)p'$ and $w = \alpha w + (1 - \alpha)w'$. Then

$$\begin{aligned} & e(p'',u') \\ & \geq \alpha e(p,u') + (1 - \alpha)e(p',u') \end{aligned}$$

$$\geq \alpha e(p,u) + (1 - \alpha)e(p',u')$$

$$= \alpha w + (1 - \alpha)w' = w'',$$

where the first inequality follows from the concavity of $e(\cdot,u)$, the second from the monotonicity of $e(\cdot,\cdot)$ in u and $u' \geq u$. We must thus have $v(p'',w'') \leq u'$.

(iv) Continuity: It is sufficient to prove the following statement. For any sequence $\{(p^n,w^n)\}_{n=1}^{\infty}$ with $(p^n,w^n) \to (p,w)$ and any u, if $v(p^n,w^n) \leq u$ for every n, then $v(p,w) \leq u$; if $v(p^n,w^n) \geq u$ for every n, then $v(p,w) \geq u$. Suppose that $v(p^n,w^n) \leq u$ for every n. Then, by the monotonicity of $e(\cdot,\cdot)$ in u and the first relation of (3.E.1), we have $w^n \leq e(p^n,u)$ for every n. By the continuity of $e(\cdot,\cdot)$, $w \leq e(p,u)$. We must thus have $v(p,w) \leq u$. The same argument can be applied for the case with $v(p^n,w^n) \geq u$ for every n.

An alternative, simpler way to show the equivalence on the concavity/ quasiconvexity and the continuity uses what is sometimes called the epigraph.

For the concavity/quasiconvexity, the concavity of $e(\cdot,u)$ is equivalent to the convexity of the set $\{(p,w): e(p,u) \geq w\}$ and the quasi-convexity of $v(\cdot)$ is equivalent to the convexity of the set $\{(p,w): v(p,w) \leq u\}$ for every u. But (3.E.1) and the monotonicity imply that $v(p,w) \leq u$ if and only if $e(p,u) \geq w$. Hence the two sets coincide and the quasiconvexity of $v(\cdot)$ is equivalent to the concavity of $e(\cdot,u)$.

As for the continuity, the function $e(\cdot)$ is continuous if and only if both $\{(p,w,u): e(p,u) \leq w\}$ and $\{(p,w,u): e(p,u) \geq w\}$ are closed sets. The function $v(\cdot)$ is continuous if and only if both $\{(p,w,u): v(p,w) \geq u\}$ and $\{(p,w,u): v(p,w) \leq u\}$ are closed sets. But, again by (3.E.1) and the monotonicity,

$$\{(p,w,u)\colon e(p,u) \le w\} = \{(p,w,u)\colon v(p,w) \ge u\};$$

$$\{(p,w,u)\colon e(p,u) \ge w\} = \{(p,w,u)\colon v(p,w) \le u\}.$$

Hence the continuity of $e(\cdot)$ is equivalent to that of $v(\cdot)$.

3.E.10 [<u>First printing errata</u>: Proposition 3.E.4 should be Proposition 3.E.3.]

Let's first prove that Proposition 3.D.2 implies Proposition 3.E.3 via the relations of (3.E.1) and (3.E.4). Let $p \in \mathbb{R}^L_{++}$ and $u \in \mathbb{R}$.

(i) Homogeneity: Let $\alpha > 0$. Define $w = e(p,u)$, then $u = v(p,w)$ by the second relation of (3.E.1). Hence

$$h(\alpha p,u) = x(\alpha p,e(\alpha p,u)) = x(\alpha p,\alpha e(p,u)) = x(p,e(p,u)) = h(p,u),$$

where the first equality follows from by the first relation of (3.E.4), the second from the homogeneity of $e(\cdot,u)$, the third from the homogeneity of $x(\cdot,\cdot)$, and the last from by the first relation of (3.E.4).

(ii) No excess utility: Let (p,u) be given and $x \in h(p,u)$. Then $x \in x(p,e(p,u))$ by the first relation of (3.E.4). Thus $u(x) = v(p,e(p,u)) = u$ by the second relation of (3.E.1).

(iii) Convexity/Uniqueness: Obvious.

Let's first prove that Proposition 3.E.3 implies Proposition 3.D.2. via the relations of (3.E.1) and (3.E.4). Let $p \in \mathbb{R}^L_{++}$ and $w \in \mathbb{R}$.

(i) Homogeneity: Let $\alpha > 0$ and define $w = e(p,u)$, then $v(p,w) = u$. Hence

$$x(\alpha p,\alpha w) = h(\alpha p,v(\alpha p,\alpha w)) = h(\alpha p,v(p,w)) = h(p,v(p,w)) = x(p,w),$$

where the first equality follows from the second relation of (3.E.4), the second from the homogeneity of $v(\cdot)$, the third from the homogeneity of $h(\cdot)$ in p, and the last from the first relation of (3.E.4).

(ii) Walras' law: Let (p,w) be given and $x \in x(p,w)$. Then $x \in h(p,v(p,w))$ by the second relation of (3.E.4). Thus $p \cdot x = e(p, v(p,w)) = w$ by the definition of the Hicksian demand and the first relation of (3.E.1).

(iii) Convexity/Uniqueness: Obvious.

3.F.1 Denote by A the intersection of the half spaces that includes K, then clearly $A \supset K$. To show the inverse inclusion, let $\bar{x} \notin K$, then, since K is a closed convex set, the separating hyperplane theorem (Theorem M.G.2) implies that there exists a $p \neq 0$ and c, such that $p \cdot \bar{x} < c < p \cdot x$ for every $x \in K$. Then $\{z \in \mathbb{R}^L : p \cdot z \geq c\}$ is a half space that includes K but does not contain x. Hence $\bar{x} \notin A$. Thus $K \supset A$.

3.F.2 If K is not a convex set, then there exists $x \in K$ and $y \in K$ such that $(1/2)x + (1/2)y \notin K$, as depicted in the figure below. The intersection of all the half-spaces containing K (which also means containing x and y) will contain the point $(1/2)x + (1/2)y$, since half-spaces are convex and the intersection of convex sets is convex. Therefore, the point $(1/2)x + (1/2)y$ cannot be separated from K.

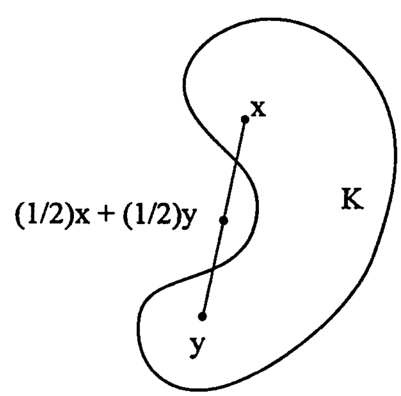


Figure 3.F.2

As for the second statement, if K is not convex, then there exist $x \in K$, $y \in K$, and $\alpha \in [0,1]$ such that $\alpha x + (1 - \alpha)y \notin K$. Since every half space that includes K also contains $\alpha x + (1 - \alpha)y$, it cannot be separated from K.

3.G.1 Since the identity $v(p,e(p,u)) = u$ holds for all p, differentiation with respect to p yields

$$\nabla_p v(p,e(p,u)) + (\partial v(p,e(p,u))/\partial w)\nabla_p e(p,u) = 0.$$

By Roy's identity,

$$(\partial v(p,e(p,u))/\partial w)(- x(p,e(p,u)) + \nabla_p e(p,u)) = 0.$$

By $\partial v(p,e(p,u))/\partial w > 0$ and $h(p,u) = x(p,e(p,u))$, we obtain $h(p,u) = \nabla_p e(p,u)$.

3.G.2 From Examples 3.D.1 and 3.E.1, for the utility function $u(x) = x_1^{\alpha}x_2^{1-\alpha}$, we obtain

$$D_w x(p,w) = \begin{bmatrix} \alpha/p_1 \\ (1 - \alpha)/p_2 \end{bmatrix},$$

$$D_p x(p,w) = \begin{bmatrix} - \alpha w/p_1^2 & 0 \\ 0 & - (1 - \alpha)w/p_2^2 \end{bmatrix},$$

$$\nabla e(p,u) = u(p_1/\alpha)^{\alpha}(p_2/(1 - \alpha))^{1-\alpha} \begin{bmatrix} \alpha/p_1 \\ (1 - \alpha)/p_2 \end{bmatrix},$$

$$\begin{aligned} D_p e(p,u) &= D_p h(p,u) \\ &= u(p_1/\alpha)^{\alpha}(p_2/(1 - \alpha))^{1-\alpha} \begin{bmatrix} - \alpha(1 - \alpha)/p_1^2 & \alpha(1 - \alpha)/p_1p_2 \\ \alpha(1 - \alpha)/p_1p_2 & - \alpha(1 - \alpha)/p_2^2 \end{bmatrix}. \end{aligned}$$

The indirect utility function for $u(x) = x_1^{\alpha}x_2^{1-\alpha}$ is

$$v(p,w) = (p_1/\alpha)^{\alpha}(p_2/(1 - \alpha))^{1-\alpha}w.$$

(Note here that the indirect utility function obtained in Example 3.D.2 is for the utility function $u(x) = \alpha \ln x_1 + (1 - \alpha)\ln x_2$.) Thus

$$\nabla_p v(p,w) = v(p,w)(- \alpha/p_1, - (1 - \alpha)/p_2),$$

$$\nabla_w v(p,w) = v(p,w)/w.$$

Hence:

$$h(p,u) = \nabla_p e(p,u),$$

$D_p^2 e(p,u) = D_p h(p,u)$, which is negative semidefinite and symmetric,

$$D_p h(p,u)p = 0,$$

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w)x(p,w)^T,$$

$$x_\ell(p,w) = -(\partial v(p,u)/\partial p_\ell)/(\partial v(p,u)/\partial w).$$

3.G.3 **(a)** Suppose that $\alpha + \beta + \gamma = 1$. Note that

$$\ln u(x) = \alpha\ln(x_1 - b_1) + \beta\ln(x_2 - b_2) + \gamma\ln(x_3 - b_3).$$

By the first-order condition of the EMP,

$$h(p,u) = (b_1,b_2,b_3) + u(p_1/\alpha)^\alpha(p_2/\beta)^\beta(p_3/\gamma)^\gamma(\alpha/p_1,\beta/p_2,\gamma/p_3).$$

Plug this into $p\cdot h(p,u)$, then we obtain the expenditure function

$$e(p,u) = p\cdot b + u(p_1/\alpha)^\alpha(p_2/\beta)^\beta(p_3/\gamma)^\gamma,$$

where $p\cdot b = p_1b_1 + p_2b_2 + p_3b_3$.

To check the homogeneity of the expenditure function,

$$e(\lambda p,u) = \lambda p\cdot b + u(\lambda p_1/\alpha)^\alpha(\lambda p_2/\beta)^\beta(\lambda p_3/\gamma)^\gamma$$
$$= \lambda p\cdot b + \lambda u(p_1/\alpha)^\alpha(p_2/\beta)^\beta(p_3/\gamma)^\gamma = \lambda e(p,u).$$

To check the monotonicity, assume $b_1 \ge 0$, $b_2 \ge 0$, and $b_3 \ge 0$. Then

$$\partial e(p,u)/\partial u = (p_1/\alpha)^\alpha(p_2/\beta)^\beta(p_3/\gamma)^\gamma > 0,$$

$$\partial e(p,u)/\partial p_1 = b_1 + u(p_1/\alpha)^\alpha(p_2/\beta)^\beta(p_3/\gamma)^\gamma(\alpha/p_1) > 0,$$

$$\partial e(p,u)/\partial p_2 = b_2 + u(p_1/\alpha)^\alpha(p_2/\beta)^\beta(p_3/\gamma)^\gamma(\beta/p_2) > 0,$$

$$\partial e(p,u)/\partial p_3 = b_3 + u(p_1/\alpha)^\alpha(p_2/\beta)^\beta(p_3/\gamma)^\gamma(\gamma/p_3) > 0.$$

To check the concavity, we can show that $D_p^2 e(p,u)$ is equal to

$$u(p_1/\alpha)^{\alpha}(p_2/\beta)^{\beta}(p_3/\gamma)^{\gamma}\begin{bmatrix} -\alpha(1-\alpha)/p_1^2 & \alpha\beta/p_1p_2 & \alpha\gamma/p_1p_3 \\ \alpha\beta/p_1p_2 & -\beta(1-\beta)/p_2^2 & \beta\gamma/p_2p_3 \\ \alpha\gamma/p_1p_3 & \beta\gamma/p_2p_3 & -\gamma(1-\gamma)/p_3^2 \end{bmatrix}.$$

and then apply the condition in Exercise 2.F.9 to show that $D_p^2e(p,u)$ is negative semidefinite. An alternative way is to only calculate the 2 × 2 submatrix obtained from $D_p^2e(p,u)$ by deleting the last row and the last column and apply the condition in Exercise 2.F.9 to show that it is negative definite. Then note that the homogeneity implies that $D_p^2e(p,u)p = 0$. Hence we can apply Theorem M.D.4(iii) to conclude that $D_p^2e(p,u)$ is negative semidefinite. The continuity follows from the functional form.

To check the homogeneity of the Hicksian demand function,

$$\begin{aligned} h(\lambda p,u) &= (b_1,b_2,b_3) + u(\lambda p_1/\alpha)^{\alpha}(\lambda p_2/\beta)^{\beta}(\lambda p_3/\gamma)^{\gamma}(\alpha/\lambda p_1,\beta/\lambda p_2,\gamma/\lambda p_3) \\ &= (b_1,b_2,b_3) + u\lambda^{\alpha+\beta+\gamma-1}(p_1/\alpha)^{\alpha}(p_2/\beta)^{\beta}(p_3/\gamma)^{\gamma}(\alpha/p_1,\beta/p_2,\gamma/p_3) \\ &= h(p,u). \end{aligned}$$

To check no excess utility,

$$u(h(p,u)) = u(p_1/\alpha)^{\alpha}(p_2/\beta)^{\beta}(p_3/\gamma)^{\gamma}(\alpha/p_1)^{\alpha}(\beta/p_2)^{\beta}(\gamma/p_3)^{\gamma} = u.$$

The uniqueness is obvious.

(b) We calculated the derivatives $\partial e(p,u)/\partial p_\ell$ in (a). If we compare them with $h_\ell(p,u)$, then we can immediately see $\partial e(p,u)/\partial p_\ell = h_\ell(p,u)$.

(c) By (b), $D_p h(p,u) = D_p^2 e(p,u)$. In (a), we calculated $D_p^2e(p,u)$. In Exercise 3.D.6, we showed

$$x(p,w) = (b_1,b_2,b_3) + (w - p\cdot b)(\alpha/p_1,\beta/p_2,\gamma/p_3)$$

and hence $D_w x(p,w) = (\alpha/p_1,\beta/p_2,\gamma/p_3)$ and

$$D_p x(p,w) = -(w - p\cdot b)\begin{bmatrix} \alpha/p_1^2 & 0 & 0 \\ 0 & \beta/p_2^2 & 0 \\ 0 & 0 & \gamma/p_3^2 \end{bmatrix} - \begin{bmatrix} \alpha/p_1 \\ \beta/p_2 \\ \gamma/p_3 \end{bmatrix}(b_1,b_2,b_3).$$

Using these results, we can verify the Slutsky equation.

(d) Use $D_p h(p,u) = D_p^2 e(p,u)$ and the explicit expression of $D_p^2 e(p,u)$ in (a).

(e) This follows from $S(p,u) = D_p h(p,u) = D_p^2 e(p,u)$ and (a), in which we showed that $D_p^2 e(p,u)$ is negative semidefinite and has rank 2.

3.G.4 **(a)** Let $a > 0$ and $b \in \mathbb{R}$. Define $\tilde{u}: \mathbb{R}_+^L \to \mathbb{R}$ by $\tilde{u}(x) = au(x) + b$ and, for each ℓ, $\tilde{u}_\ell: \mathbb{R}_+ \to \mathbb{R}$ by $\tilde{u}_\ell(x_\ell) = au_\ell(x_\ell) + b/L$. Then

$$\tilde{u}(x) = a\textstyle\sum_\ell u_\ell(x_\ell) + b = \sum_\ell(au_\ell(x_\ell) + b/L) = \sum_\ell \tilde{u}_\ell(x_\ell).$$

Thus any linear (to be exact, affine) transformation of a separable utility function is again separable.

Next, we prove that if a monotone transformation of a separable utility function is again separable, then the monotone transformation must be linear (affine). To do this, let's assume that each $u_\ell(\cdot)$ is continuous and strongly monotone. Then, for each ℓ, the range $u_\ell(\mathbb{R}_+)$ is a half-open interval. So let $u_\ell(\mathbb{R}_+) = [a_\ell,b_\ell)$, where b_ℓ may be $+\infty$. Define $c_\ell = b_\ell - a_\ell > 0$, $a = \sum_\ell a_\ell$, $b = \sum_\ell b_\ell$, and $c = \sum_\ell c_\ell$. (If some b_ℓ is equal to $+\infty$, then b and c are $+\infty$ as well.) Then $u(\mathbb{R}_+^L) = [a,b)$. Suppose that $f: [a,b) \to \mathbb{R}$ is strongly monotone and the utility function $\tilde{u}(\cdot)$ defined by $\tilde{u}(x) = f(u(x))$ is separable. To simplify the proof, let's assume that $f(\cdot)$ is differentiable. Define $g: [0,c) \to \mathbb{R}$ by

$$g(v) = f(v + a) - f(a),$$

then $g(0) = 0$, $g(\cdot)$ is differentiable, and

$$g(u(x) - u(0)) = f(u(x)) - f(u(0))$$

for every $x \in \mathbb{R}^L_+$. Thus, in order to prove that $f(\cdot)$ is linear (affine), it is sufficient to prove that $g(\cdot)$ is linear. For this, it is sufficient to show that the first-order derivatives $g'(v)$ do not depend on the choice of $v \in [0,c)$.

To this end, we shall first prove that if $v_\ell \in [0,c_\ell)$ for each ℓ, then $g(\sum_\ell v_\ell) = \sum_\ell g(v_\ell)$. For this, it is sufficient to prove that

$$g(u(x) - u(0)) = \sum_\ell g(u_\ell(x_\ell) - u_\ell(0))$$

for every $x \in \mathbb{R}^L_+$. In fact, by the separability assumption, for each ℓ, there exists a monotone utility function $\tilde{u}_\ell(\cdot)$ such that $\tilde{u}(x) = \sum_\ell \tilde{u}_\ell(x_\ell)$ for every $x \in \mathbb{R}^L_+$. Fix an $x \in \mathbb{R}^L_+$ and, for each ℓ, define $y^\ell \in \mathbb{R}^L_+$ by $y^\ell_\ell = x_\ell$ and $y^\ell_k = 0$ for any $k \neq \ell$. Since $\tilde{u}(y^\ell) = f(u(y^\ell))$,

$$\tilde{u}_\ell(x_\ell) + \sum_{k\neq\ell}\tilde{u}_k(0) = f(u_\ell(x_\ell) + \sum_{k\neq\ell}u_k(0)).$$

Subtracting $\sum_{k=1}^L \tilde{u}_k(0) = \tilde{u}(0) = f(u(0))$ from both sides and noticing that $u_\ell(x_\ell) + \sum_{k\neq\ell}u_k(0) = u_\ell(x_\ell) - u_\ell(0) + \sum_{k=1}^L u_k(0)$, we obtain

$$\tilde{u}_\ell(x_\ell) - \tilde{u}_\ell(0) = g(u_\ell(x_\ell) - u_\ell(0)).$$

Summing over ℓ, we obtain

$$\sum_\ell\tilde{u}_\ell(x_\ell) - \sum_\ell\tilde{u}_\ell(0) = \sum_\ell g(u_\ell(x_\ell) - u_\ell(0)).$$

Since

$$\sum_\ell\tilde{u}_\ell(x_\ell) - \sum_\ell\tilde{u}_\ell(0) = \tilde{u}(x) - \tilde{u}(0) = f(u(x)) - f(u(0)) = g(u(x) - u(0)),$$

we have

$$g(u(x) - u(0)) = \sum_\ell g(u_\ell(x_\ell) - u(0)).$$

We have thus proved that $g(\sum_\ell v_\ell) = \sum_\ell g(v_\ell)$.

To prove that the $g'(v)$ do not depend on the choice of $v \in [0,c)$, note first that if $v_\ell \in [0,c_\ell)$ for each ℓ and $v = \sum_\ell v_\ell \in [0,c)$, then $g'(v) = g'(v_\ell)$

for each ℓ. This can be established by differentiating both sides of $g(\sum_\ell v_\ell) = \sum_\ell g(v_\ell)$ with respect to v_ℓ.

So let $v \in [0,c)$ and $v' \in [0,c)$, then, for each ℓ, there exist $v_\ell \in [0,c_\ell)$ and $v'_\ell \in [0,c_\ell)$ such that $v = \sum_\ell v_\ell$ and $v' = \sum_\ell v'_\ell$. Then $g'(v) = g'(v_1)$ and $g'(v') = g'(v'_1)$. Now, for some $\bar{v}_2 \in [0,c_2)$, consider $v_1 + \bar{v}_2 \in [0,c)$ and $v'_1 + \bar{v}_2 \in [0,c)$. Then

$$g'(v_1) = g'(v_1 + \bar{v}_2) = g'(\bar{v}_2),$$
$$g'(v'_1) = g'(v'_1 + \bar{v}_2) = g'(\bar{v}_2).$$

Thus $g'(v_1) = g'(v'_1)$ and hence $g'(v) = g'(v')$.

Note that the above proof by means of derivatives is underlain by the cardinal property of additively separable utility function, which is that, when moving from one commodity vector to another, if the loss in utility from some commodity is exactly compensated by the gain in utility from another, then this must be the case for any of its monotone transformations resulting in another additively separable utility function. (This fact is often much more shortly put into as: utility differences matter.) For example, consider $x \in \mathbb{R}^L_+$ and $x' \in \mathbb{R}^L_+$ such that $u_1(x_1) - u_1(x'_1) = u_2(x'_2) - u_2(x_2) > 0$ and $x_\ell = x'_\ell$ for every $\ell \geq 3$. Since $u_1(x_1) + u_2(x_2) = u_1(x'_1) + u_2(x'_2)$, the separability implies that $u(x) = u(x')$. By the equality $g(\sum_\ell v_\ell) = \sum_\ell g(v_\ell)$,

$$g(u_1(x_1) - u_1(0)) + g(u_2(x_2) - u_2(0)) = g(u_1(x'_1) - u_1(0)) + g(u_2(x'_2) - u_2(0)).$$

Hence

$$g(u_1(x_1) - u_1(0)) - g(u_1(x'_1) - u_1(0)) = g(u_2(x'_2) - u_2(0)) - g(u_2(x_2) - u_2(0)).$$

We have shown that, under the differentiability assumption, if this holds for every x_1, x_2, x'_1, and x'_2, then $g(\cdot)$ must be linear.

(b) Define $S = \{1,\ldots,L\}$ and let T be a subset of S. The commodity vectors for those in S are represented by $z_1 = (z_\ell)_{\ell \in T} \in \mathbb{R}_+^{\#T}$ and the like, and the commodity vectors for those outside S are represented by $z_2 = (z_\ell)_{\ell \notin T} \in \mathbb{R}_+^{L-\#T}$ and the like. We shall prove that for every $z_1 \in \mathbb{R}_+^{\#T}$, $z_1' \in \mathbb{R}_+^{\#T}$, $z_2 \in \mathbb{R}_+^{L-\#T}$, and $z_2' \in \mathbb{R}_+^{L-\#T}$, $(z_1,z_2) \succsim (z_1',z_2)$ if and only if $(z_1,z_2') \succsim (z_1',z_2')$. In fact, since $u(\cdot)$ represents $\succsim$, $(z_1,z_2) \succsim (z_1',z_2)$ if and only if

$$\Sigma_{\ell \in T} u_\ell(z_\ell) + \Sigma_{\ell \notin T} u_\ell(z_\ell) \geq \Sigma_{\ell \in T} u_\ell(z_\ell') + \Sigma_{\ell \notin T} u_\ell(z_\ell).$$

Likewise, $(z_1,z_2') \succsim (z_1',z_2')$ if and only if

$$\Sigma_{\ell \in T} u_\ell(z_\ell) + \Sigma_{\ell \notin T} u_\ell(z_\ell') \geq \Sigma_{\ell \in T} u_\ell(z_\ell') + \Sigma_{\ell \notin T} u_\ell(z_\ell').$$

But both of these two inequalities are equivalent to

$$\Sigma_{\ell \in T} u_\ell(z_\ell) \geq \Sigma_{\ell \in T} u_\ell(z_\ell').$$

Hence they are equivalent to each other.

(c) Suppose that the wealth level w increases and all prices remain unchanged. Then the demand for at least one good (say, good ℓ) has to increase by the Walras' law. From (3.D.4) we know that $u_k'(x_k(p,w)) = (p_k/p_\ell)u_\ell'(x_\ell(p,w))$ for every $k = 1,\ldots,L$. Since $x_\ell(p,w)$ increased and $u_\ell(\cdot)$ is strictly concave, the right hand side will decrease. Hence, again since $u_k(\cdot)$ is strictly concave, $x_k(p,w)$ will have to increase. Thus all goods are normal.

(d) The first-order condition of the UMP can be written as

$$\lambda(p,w)p_\ell = \hat{u}'(x_\ell(p,w)),$$

where the Lagrange multiplier $\lambda(p,w)$ is a differentiable function of (p,w): This can be easily seen in the proof of the differentiability of Walrasian demand functions, which is contained in the Appendix.

By differentiating the above first-order condition with respect to p_ℓ, we obtain

$$(\partial\lambda(p,w)/\partial p_\ell)p_\ell + \lambda(p,w) = \hat{u}''(x_\ell(p,w))(\partial x_\ell(p,w)/\partial p_\ell).$$

By differentiating the above first-order condition with respect to p_k ($k \neq \ell$), we obtain

$$(*) \qquad (\partial\lambda(p,w)/\partial p_k)p_\ell = \hat{u}''(x_\ell(p,w))(\partial x_\ell(p,w)/\partial p_k).$$

Thus

$$\begin{aligned}
& d[p\cdot x(p,w)]/dp_k \\
&= d[\textstyle\sum_\ell p_\ell x_\ell(p,w)]/dp_k \\
&= x_k(p,w) + p_k(\partial x_k(p,w)/\partial p_k) + \textstyle\sum_{\ell\neq k} p_\ell(\partial x_\ell(p,w)/p_k). \\
&= x_k(p,w) + \frac{(\partial\lambda(p,w)/\partial p_k)p_k^2 + \lambda(p,w)p_k}{\hat{u}''(x_k(p,w))} + \sum_{\ell\neq k}\frac{(\partial\lambda(p,w)/\partial p_k)p_\ell^2}{\hat{u}''(x_\ell(p,w))} \\
&= x_k(p,w) + \frac{\lambda(p,w)p_k}{\hat{u}''(x_k(p,w))} + (\partial\lambda(p,w)/\partial p_k)\sum_\ell \frac{p_\ell^2}{\hat{u}''(x_\ell(p,w))}.
\end{aligned}$$

By the first-order condition, $\lambda(p,w)p_k = \hat{u}'(x_k(p,w))$ and hence this equals

$$\frac{\hat{u}'(x_k(p,w))}{\hat{u}''(x_k(p,w))}\left(\frac{x_k(p,w)\hat{u}'(x_k(p,w))}{\hat{u}''(x_k(p,w))} + 1\right) + (\partial\lambda(p,w)/\partial p_k)\sum_\ell \frac{p_\ell^2}{\hat{u}''(x_\ell(p,w))}.$$

By Walras' law, this equals zero. By the strong monotonicity and the strict concavity, $\dfrac{\hat{u}'(x_k(p,w))}{\hat{u}''(x_k(p,w))} < 0$ and $\sum_\ell \dfrac{p_\ell^2}{\hat{u}''(x_\ell(p,w))} < 0$. By the assumption on $\hat{u}(\cdot)$, $\dfrac{x_k(p,w)\hat{u}'(x_k(p,w))}{\hat{u}''(x_k(p,w))} + 1 > 0$. Hence

$$\frac{\hat{u}'(x_k(p,w))}{\hat{u}''(x_k(p,w))}\left(\frac{x_k(p,w)\hat{u}'(x_k(p,w))}{\hat{u}''(x_k(p,w))} + 1\right) < 0.$$

We must thus have $\partial\lambda(p,w)/\partial p_k < 0$. Hence, by (*), $\partial x_\ell(p,w)/\partial p_k > 0$.

3.G.5 **(a)** We shall show the following two statements: First, if (x^*,z^*) is a solution to

$$\text{Max}_{(x,z)}\ \tilde{u}(x,z) \quad \text{s.t. } p\cdot x + \alpha z \le w,$$

then there exists y* such that $q\cdot y^* \le z^*$ and (x^*,y^*) is a solution to

$$\text{Max}_{(x,y)}\ u(x,y) \quad \text{s.t. } p\cdot x + (\alpha q_0)\cdot y \le w;$$

second, if (x^*,y^*) is a solution to

$$\text{Max}_{(x,y)}\ u(x,y) \quad \text{s.t. } p\cdot x + (\alpha q_0)\cdot y \le w,$$

then $(x^*,q_0\cdot y^*)$ is a solution to

$$\text{Max}_{(x,z)}\ \tilde{u}(x,z) \quad \text{s.t. } p\cdot x + \alpha z \le w.$$

Suppose first that (x^*,z^*) is a solution to $\text{Max}_{(x,z)}\ \tilde{u}(x,z)$ s.t. $p\cdot x + \alpha z \le w$. Then, by the definition of $\tilde{u}(\cdot)$, there exists y* such that $q_0\cdot y^* \le z^*$ and $u(x^*,y^*) = \tilde{u}(x^*,z^*)$. Let (x,y) satisfy $p\cdot x + (\alpha q_0)\cdot y \le w$. Then

$$u(x,y) \le \tilde{u}(x,q_0\cdot y) \le u(x^*,z^*) = u(x^*,y^*),$$

where the first inequality follows from the definition of $\tilde{u}(\cdot)$ and the second inequality follows from $p\cdot x + \alpha(q_0\cdot y) = p\cdot x + (\alpha q_0)\cdot y \le w$ and the definition of (x^*,z^*). The first statement is thus established.

As for the second one, suppose that (x^*,y^*) is a solution to $\text{Max}_{(x,y)}$ $u(x,y)$ s.t. $p\cdot x + (\alpha q_0)\cdot y \le w$. For every y, if $q_0\cdot y \le q_0\cdot y^*$, then

$$p\cdot x^* + (\alpha q_0)\cdot y = p\cdot x^* + \alpha(q_0\cdot y) \le p\cdot x^* + \alpha(q_0\cdot y^*) \le w.$$

Hence $u(x^*,y) \le u(x^*,y^*)$. Thus $u(x^*,y^*) = \tilde{u}(x^*,q_0\cdot y^*)$. Now let (x,z) satisfy $p\cdot x + \alpha z \le w$. Then there exists y such that $q_0\cdot y \le z$ and $u(x,y) = \tilde{u}(x,z)$. Thus $p\cdot x + (\alpha q_0)\cdot y = p\cdot x + \alpha(q_0\cdot y) \le p\cdot x + \alpha z \le w$. Hence

$$\tilde{u}(x,z) = u(x,y) \le u(x^*,y^*) = \tilde{u}(x^*,q_0\cdot y^*).$$

(b) (c) These are immediate consequences of the fact that the Walrasian demand functions and the Hicksian demand functions are derived in the standard way, by taking $\tilde{u}(\cdot)$ to be the (primitive) direct utility function.

3.G.6 **(a)** By applying Walras' law, we obtain $x_3 = (w - x_1p_1 - x_2p_2)/p_3$.

(b) Yes. In fact, for every $\lambda > 0$,

$$100 - 5\lambda p_1/\lambda p_3 + \beta\lambda p_2/\lambda p_3 + \delta\lambda w/\lambda p_3 = 100 - 5p_1/p_3 + \beta p_2/p_3 + \delta w/p_3,$$

$$\alpha + \beta\lambda p_1/\lambda p_3 + \gamma\lambda p_2/\lambda p_3 + \delta\lambda w/\lambda p_3 = \alpha + \beta p_1/p_3 + \gamma p_2/p_3 + \delta w/p_3.$$

(c) By Proposition 3.G.2 and 3.G.3, the Slutsky substitution matrix is symmetric. Thus

$$\beta/p_3 + (\delta/p_3)(\alpha + \beta p_1/p_3 + \gamma p_2/p_3 + \delta w/p_3)$$
$$= \beta/p_3 + (\delta/p_3)(100 - 5p_1/p_3 + \beta p_2/p_3 + \delta w/p_3).$$

Hence by putting $p_3 = 1$ and rearranging terms, we obtain

$$(\beta + \alpha\delta) + \beta\delta p_1 + \gamma\delta p_2 + \delta^2 w = (\beta + 100\delta) - 5\delta p_1 + \beta\delta p_2 + \delta^2 w.$$

Since this equality must hold for all p_1, p_2 and w, we have

$$\beta + \alpha\delta = \beta + 100\delta,\ \beta\delta = -5\delta,\ \gamma\delta = \beta\delta.$$

Hence $\alpha = 100$, $\beta = -5$, and $\gamma = -5$. Therefore,

$$x_1 = x_2 = 100 - 5p_1/p_3 + 5p_2/p_3 + \delta w/p_3.$$

Recall also that all diagonal entries of the Slutsky matrix must be nonpositive. We shall now derive from this fact that $\delta = 0$. Let $p_3 = 1$, then the first diagonal element is equal to

$$-5 + \delta(100 - 5p_1 + 5p_2) + \delta^2 w.$$

If $\delta \neq 0$, then $\delta^2 > 0$ and hence we can always find (p_1,p_2,w) such that the above value is positive. We must thus have $\delta = 0$. In conclusion,

$$x_1 = x_2 = 100 - 5p_1/p_3 + 5p_2/p_3.$$

(d) Since $x_1 = x_2$ for all prices, the consumer's indifference curves in the (x_1,x_2)-plane must be L-shaped ones, with kinks on the diagonal. (So the restricted preference is the Leontief one.) They are depicted in the following figure.

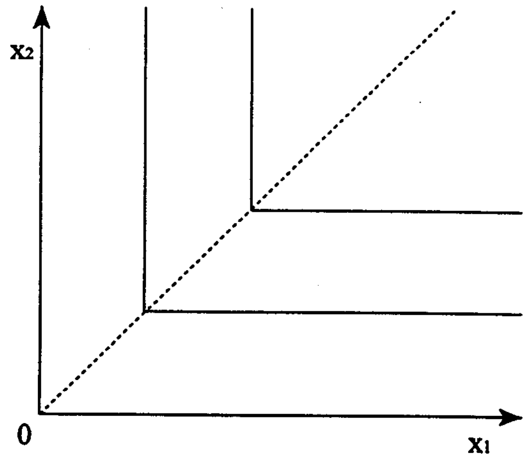


Figure 3.G.6(d)

(e) By (d), for a fixed x_3, the preference for goods 1 and 2 can be represented by $\min\{x_1,x_2\}$. Moreover, there is no wealth effect on the demands for goods 1 and 2. We must thus have

$$u(x_1,x_2,x_3) = \min\{x_1,x_2\} + x_3$$

or a monotone transformation of this.

3.G.7 By the first-order condition of the UMP, there exists $\lambda > 0$ such that $\lambda g(x) = \nabla u(x)$. Premultiply both sides by x, then $\lambda x \cdot g(x) = x \cdot \nabla u(x)$. By Walras' law, $x \cdot g(x) = 1$ and hence $\lambda = x \cdot \nabla u(x)$. Thus

$$g(x) = \lambda^{-1} \nabla u(x) = \frac{1}{x \cdot \nabla u(x)} \nabla u(x)$$

By Exercise 3.D.8, we have $\partial v(p,1)/\partial w = -p \cdot \nabla_p v(p,1)$. By Proposition 3.G.4, $x(p) = -\frac{1}{\partial v(p,1)/\partial w} \nabla_p v(p,1) = \frac{1}{p \cdot \nabla_p v(p,1)} \nabla_p v(p,1)$.

3.G.8 Differentiate the equality $v(p,\alpha w) = v(p,w) + \ln\alpha$ with respect to α and evaluate at $\alpha = 1$, then we obtain $(\partial v(p,w)/\partial w)w = 1$. Hence $\partial v(p,1)/\partial w = 1$. By Proposition 3.G.4, $x(p,1) = -\nabla_p v(p,1)$.

3.G.9 Let $p \gg 0$ and $w > 0$. All the functions and derivatives below are evaluated at (p,w).

By differentiating Roy's identity with respect to p_k, we obtain

$$\partial x_\ell/\partial p_k = \frac{(\partial^2 v/\partial p_\ell \partial p_k)(\partial v/\partial w) - (\partial v/\partial p_\ell)(\partial^2 v/\partial p_\ell \partial w)}{(\partial v/\partial w)^2}.$$

Or, in the matrix notation,

$$D_p x = -\frac{1}{(\nabla_w v)^2}(\nabla_w v D_p^2 v - \nabla_p v D_p \nabla_w v) \in \mathbb{R}^{L\times L}.$$

(Recall that $\nabla_p v$ is a column vector of $\mathbb{R}^L$, and $D_p v$ and $D_p \nabla_w v$ are row vectors of $\mathbb{R}^L$ (Section M.A).) By differentiating Roy's identity with respect to w, we obtain

$$\partial x_\ell/\partial w = \frac{(\partial^2 v/\partial p_\ell \partial w)(\partial v/\partial w) - (\partial v/\partial p_\ell)(\partial^2 v/\partial w^2)}{(\partial v/\partial w)^2}.$$

Or, in matrix notation,

$$D_w x = -\frac{1}{(\nabla_w v)^2}(\nabla_w v \nabla_p \nabla_w v - \nabla_w^2 v \nabla_p v) \in \mathbb{R}^L.$$

Hence

$$S = -\frac{1}{(\nabla_w v)^2}(\nabla_w v D_p^2 v - \nabla_p v D_p \nabla_w v - (\nabla_w v \nabla_p \nabla_w v)(\frac{1}{\nabla_w v} D_p v) + \nabla_w^2 v \ \nabla_p v(\frac{1}{\nabla_w v} D_p v))$$

$$= -\frac{1}{(\nabla_w v)^2}(\nabla_w v D_p^2 v - (\nabla_p v D_p \nabla_w v + \nabla_p \nabla_w v D_p v) + \frac{\nabla_w^2 v}{\nabla_w v} \nabla_p v D_p v).$$

It is noteworthy that we can know directly from Roy's identity and this equality that the Slutsky matrix S has all the properties stated in Proposition 3.G.2. To see this, note first that both $\nabla_p v(p,w)$ and $\nabla_w v(p,w)$ are homogeneous of degree -1 (in (p,w)) by Theorem M.B.1. Hence $x(p,w)$ is homogeneous of degree 0 (where we are regarding Roy's identity as defining $x(p,w)$ from $\nabla_p v(p,w)$ and $\nabla_w v(p,w)$). Thus (2.E.2) follows, as proved in Proposition 2.E.1. On the other hand, by Exercise 3.D.8, $p\cdot x(p,w) = w$.

Hence, as proved in Propositions 2.E.2 and 2.E.3, this implies (2.E.5) and (2.E.7). Now, as proved in Exercise 2.F.7, (2.E.2), (2.E.5), and (2.E.7) together imply that $S(p,w)p = 0$ and $p\cdot S(p,w) = 0$.

The matrix $S(p,w)$ is symmetric because $D_p^2 v$, $\nabla_p v D_p \nabla_w v + \nabla_p \nabla_w v D_p v$, and $\nabla_p v D_p v$ are symmetric. The negative semidefiniteness can be shown in the following way. Since $v(\cdot)$ is quasiconvex, for every price-wealth pair (q,b), if $D_p vq + \nabla_w vb = 0$, then $(q,b)\cdot D^2 v\,(q,b) \geq 0$ (Theorem M.C.4). But $D_p vq + \nabla_w vb = 0$ if and only if $b = -D_p vq/\nabla_w v$. Plug this into $(q,b)\cdot D^2 v(q,b)$, then

$$\begin{aligned}
&q\cdot D_p^2 vq + 2b(D_p \nabla_w vq) + \nabla_w^2 v\, b^2 \\
&= q\cdot D_p^2 vq - \frac{2D_p \nabla_w vq}{\nabla_w v} D_p vq + \nabla_w^2 v \frac{(D_p vq)^2}{(\nabla_w v)^2} \\
&= \frac{1}{\nabla_w v}\Big((\nabla_w v)(q\cdot D_p^2 vq) - 2(D_p \nabla_w vq)(D_p vq) + \nabla_w^2 v \frac{(D_p vq)^2}{(\nabla_p v)^2}\Big).
\end{aligned}$$

Hence the quasiconvexity of $v(\cdot)$ implies that the above expression is nonnegative for every q. On the other hand,

$$q\cdot S(p,w)q = -\frac{1}{(\nabla_w v)^2}\Big((\nabla_w v)(q\cdot D_p^2 vq) - 2(D_p \nabla_w vq)(D_p vq) + \nabla_w^2 v \frac{(D_p vq)^2}{(\nabla_w v)^2}\Big).$$

Thus $q\cdot S(p,w)q \leq 0$. Therefore $S(p,w)$ is negative semidefinite.

3.G.10 We shall prove that $a(p)$ is a constant function and $b(p)$ is homogeneous of degree -1, quasiconvex, and satisfies $b(p) \geq 0$ and $\nabla b(p) \leq 0$ for every $p \gg 0$.

We shall first prove that $a(p)$ must be homogeneous of degree zero. Since $v(p,w)$ is homogeneity of degree zero, we must have $a(\lambda p) + b(\lambda p)\lambda w = a(p) + b(p)w$ for all $p \gg 0$, $w \geq 0$, and $\lambda > 0$. If there are p and λ for which $a(\lambda p) \neq a(p)$, then this constitutes an immediate contradiction with $w = 0$. Thus

$a(p)$ is homogeneous of degree zero.

Next, we show by contradiction that $\nabla a(p) \leq 0$ for every $p \gg 0$. Since $v(p,w)$ is nonincreasing in p, we must have $\nabla a(p) + \nabla b(p)w \leq 0$ for every $p \gg 0$ and $w \geq 0$. If there are $p \gg 0$ and ℓ for which $\partial a(p)/\partial p_\ell > 0$, then this constitutes an immediate contradiction with $w = 0$. Thus $a(p) \leq 0$ for every $p \gg 0$.

We shall now prove that the only function $a(p)$ that is homogeneous of degree zero and satisfies $\nabla a(p) \leq 0$ for every $p \gg 0$ is a constant function. In fact, by differentiating both sides of $a(\lambda p) = a(p)$ with respect to λ and evaluating them at $\lambda = 1$, we obtain $\nabla a(p)p = 0$. Since $p \gg 0$ and $\nabla a(p) \leq 0$, we must have $\nabla a(p) = 0$. Hence $a(p)$ must be constant.

Given this result, the homogeneity of $v(p,w)$ implies that $b(p)$ is homogeneous of degree -1. Since $v(p,w)$ is quasiconvex, so is $a(p)$ as a function of p. Finally, since $\nabla_p v(p,w) = \nabla b(p)w \leq 0$ and $\nabla_w v(p,w) = b(p)$, we must have $b(p) \geq 0$ and $\nabla b(p) \leq 0$ for every $p \gg 0$.

This result implies that, up to a constant, $v(p,w) = b(p)w$ and hence, if the underlying utility function is quasiconcave, then it must be homogeneous of degree one. On the other hand, according to Exercise 3.D.4(b), if the underlying utility function is quasilinear with respect to good 1, then, for all w and $p \gg 0$ with $p_1 = 1$, $v(p,w)$ can be written in the form $\phi(p_2,\dots,p_L) + w$. You will thus wonder why we have ended up excluding this quasilinear case. The reason is that, when we derived $v(p,w) = \phi(p) + w$, we assumed that the consumption set is $(-\infty, \infty) \times \mathbb{R}_+^{L-1}$. Thus $x_1(p,w)$ can be negative and, if so, then $\partial v(p,w)/\partial p_1$ is positive, which we excluded at the beginning of our analysis, following Proposition 3.D.3. (Note that, when we established $\partial v(p,w)/\partial p_\ell \leq 0$ in Proposition 3.D.3, we assumed that the consumption set is

$\mathbb{R}^L_+$.) As $x_1(p,w)$ is positive for sufficiently large $w > 0$, the quasilinear case could be accommodated in our analysis if we assume that the inequality $\partial v(p,w)/\partial p_\ell \le 0$ applies only for sufficiently large $w > 0$. (In this case, we can show that $a(p)$ is homogeneous of degree zero, and $b(p)$ is homogeneous of degree -1, quasiconvex, and satisfies $b(p) \ge 0$ and $\nabla b(p) \le 0$ for every $p \gg 0$.)

3.G.11 Suppose that $v(p,w) = a(p) + b(p)w$. By Roy's identity,

$$x(p,w) = -(1/b(p))\nabla_p a(p) - (w/b(p))\nabla_p b(p).$$

Thus the wealth expansion path is linear in the direction of $\nabla_p b(p)$ and intercept $(-1/b(p))\nabla_p a(p)$.

3.G.12 Note first that, according to Exercise 3.G.11, the wealth expansion path is linear in the direction of $\nabla_p b(p)$ and intercept $(-1/b(p))\nabla_p a(p)$.

If the underlying preference is homothetic, then $(-1/b(p))\nabla_p a(p) = 0$. Hence $a(p)$ must be a constant function. If the underlying utility function is homogeneous of degree one in w, then $v(p,w)$ must be homogeneous of degree one in w by Exercise 3.D.3(a). Hence $a(p) = 0$ for every $p \gg 0$.

If the preference is quasilinear in good 1, then please first go back to the proviso given at the end of the answer to Exercise 3.G.10. After doing so, note that, since the demand for goods 2,...,L do not depend on w,

$$(-1/b(p))\nabla_p b(p) = (-1/p_1, 0, \dots, 0),$$

or $(\partial b(p)/\partial p_1)/b(p) = 1/p_1$ and $\partial b(p)/\partial p_\ell = 0$ for every $\ell > 1$. Hence $b(p) = \beta p_1^\rho + \gamma$ for some $\beta \ne 0$, $\rho \ne 0$, and $\gamma \in \mathbb{R}$. But, by Exercise 3.G.10, $b(p)$ must be homogeneous of degree -1, positive, and nonincreasing. Hence $\rho = -1$, $\gamma = 0$, and $\beta > 0$. That is, $b(p) = \beta/p_1$ with $\beta > 0$. If the underlying utility function is in the quasilinear form $x_1 + \tilde{u}(x_2,\ldots,x_L)$, then, by Exercise 3.D.4(b), $v(p,w)$ must be written in the form $\phi(p_2,\ldots,p_L) + w$ for all $p \gg 0$ with $p_1 = 1$. Thus $\beta = 1$.

3.G.13 For each $i \in \{0,1,\ldots,n\}$, let a_i be a differentiable function defined on the strictly positive orthant $\{p \in \mathbb{R}^L: p \gg 0\}$. Let $v(p,w) = \sum_{i=0}^{n} a_i(p) w^i$ be an indirect utility function. Denoting the corresponding Walrasian demand function by $x(p,w)$. By Roy's identity,

$$x(p,w) = -\frac{1}{\nabla_w v(p,w)} \nabla_p v(p,w)$$

$$= \frac{1}{\sum_{i=1}^{n} i a_i(p) w^{i-1}} \sum_{i=0}^{n} w^i \nabla a_i(p) = \sum_{i=0}^{n} \frac{w^i}{\sum_{j=1}^{n} j a_j(p) w^{j-1}} \nabla a_i(p).$$

Hence, for any fixed p, the wealth expansion path is contained the linear subspace of $\mathbb{R}^L$ that is spanned by $\nabla a_0(p),\ldots,\nabla a_n(p)$.

As for the interpretation, recall from Exercise 3.G.11 that, an indirect utility function in the Gorman form exhibits linear wealth expansion curves. But the Gorman form is a polynomial of degree one on w and a linear wealth expansion curve is contained in a linear subspace of dimension two. Hence the above result implies that the indirect utility functions that are polynomials on w is a natural extension of the Gorman form.

3.G.14 Define a, b, c, d, e, and f so that

$$\begin{bmatrix} -10 & a & b \\ c & -4 & d \\ 3 & e & f \end{bmatrix}.$$

Since the substitution matrix is symmetric, we know b = 3, a = c, and e = d. By Propositions 2.F.3, $p \cdot S(p,w) = 0$. Hence $p_1(-10) + p_2 c + p_3 3 = 0$. Thus c = - 4 and a = c = - 4. For the second column, $p_1(-4) + p_2(-4) + p_3 e = 0$.

Hence e = 2 and d = e = 2. Finally, for the third column, we have $p_1 3 + p_2 2 + p_3 f = 0$. Thus f = - 7/6. Hence we have

$$\begin{bmatrix} -10 & -4 & 3 \\ -4 & -4 & 2 \\ 3 & 2 & -7/6 \end{bmatrix}.$$

The matrix has all the properties of a substitution matrix, which are symmetry, negative semidefiniteness, $S(p,w)p = 0$, and $p \cdot S(p,w) = 0$. (For negative semidefiniteness, apply the determinant test of Exercise 2.F.10 and Theorem M.D.4(iii).)

3.G.15 **(a)** $x(p_1,p_2,w) = \left(\dfrac{p_2 w}{p_1 p_2 + 4p_1^2}, \dfrac{4p_1 w}{4p_1 p_2 + p_2^2}\right)$.

(b) $h(p_1,p_2,u) = \left(\left(\dfrac{p_2 u}{2(4p_1 + p_2)}\right)^2, \left(\dfrac{p_1 u}{4p_1 + p_2}\right)^2\right)$.

(c) $e(p_1,p_2,u) = \dfrac{p_1 p_2 u^2}{4(4p_1 + p_2)}$. It is then easy to show that $\nabla_p e(p_1,p_2,u) = h(p_1,p_2,u)$.

(d) $v(p_1,p_2,w) = 2(w/p_1 + 4w/p_2)^{1/2}$. To verify Roy's identity, use

$$\partial v(p_1,p_2,w)/\partial p_1 = (w/p_1 + 4w/p_2)^{-1/2}(-\,w/p_1^2),$$
$$\partial v(p_1,p_2,w)/\partial p_2 = (w/p_1 + 4w/p_2)^{-1/2}(-\,4w/p_2^2),$$
$$\partial v(p_1,p_2,w)/\partial w = (w/p_1 + 4w/p_2)^{-1/2}(1/p_1 + 4/p_2).$$

3.G.16 **(a)** It is easy to check that

$$\partial e(p,u)/\partial p_k = e(p,u)(\alpha_k + u\beta_k(\Pi_\ell p_\ell^{\beta_\ell}))/p_k.$$

Since $e(p,u)$ is nondecreasing in p, this must be nonnegative for all (p,u). But, if $\alpha_k < 0$ and $\|p\|$ is sufficiently small, then this becomes negative. Also, if $\beta_k < 0$ and $\|p\|$ is sufficiently big, then this becomes negative.

Therefore

(1) $$\alpha_k \geq 0 \text{ and } \beta_k \geq 0 \text{ for all k.}$$

It is a little bit manipulation to show that

$$e(\lambda p,u) = \lambda^{\sum\alpha_\ell}\exp((\sum_\ell\alpha_\ell \ln p_\ell) + \lambda^{\sum\beta_\ell}u(\Pi_\ell p_\ell^{\beta_\ell})),$$

$$\lambda e(p,u) = \lambda\exp((\Pi_\ell\alpha_\ell \ln p_\ell) + u(\Pi_\ell p_\ell^{\beta_\ell})).$$

Since $e(p,u)$ is homogeneous of degree one with respect to p, they must be equal for every (p,u) and $\lambda > 0$. Take, for example, p = (1,...,1) and u = 1. Then

$$\log e(p,u) = (\sum_\ell\alpha_\ell)\log\lambda + \lambda^{\sum\beta_\ell},$$

$$\log \lambda e(p,u) = \log\lambda + 1.$$

They must be equal for every $\lambda > 0$. Therefore

(2) $$\sum_\ell\alpha_\ell = 1,\ \sum_\ell\beta_\ell = 0.$$

Thus

$$\sum_\ell\alpha_\ell = 1,\ \alpha_\ell \geq 0,\ \beta_\ell = 0.$$

Hence the expenditure function now takes the simplified form:

(3) $$e(p,u) = (\exp u)(\Pi_\ell p_\ell^{\alpha_\ell});\ \sum_\ell\alpha_\ell = 1,\ \alpha_\ell \geq 0.$$

This is increasing with respect to u and concave in p.

(b) By equation (3.E.1), $w = (\exp v(p,w))(\Pi_\ell p_\ell^{\alpha_\ell})$. Hence

(4) $$v(p,w) = \log w - \sum_\ell\alpha_\ell \log p_\ell.$$

(c) By differentiating $e(p,u)$ with respect to p, we obtain

$$h(p,u) = e(p,u)(\alpha_1/p_1,\ldots,\alpha_L/p_L).$$

Since $x(p,w) = h(p,v(p,w))$ and $e(p,v(p,w)) = w$,

(5) $$x(p,w) = w(\alpha_1/p_1,\ldots,\alpha_L/p_L).$$

Use equations (3), (4), and (5) and follows the same method as in the answer to Exercise 3.G.2 to verify Roy's identity and the Slutsky equation.

3.G.17 [First printing errata: The minus sign at the beginning of the right-hand side of the indirect utility function should be deleted. That is, it should be

$$v(p,w) = (w/p_2 + b^{-1}(ap_1/p_2 + a/b + c))\exp(-\ bp_1/p_2).$$

Also, in (b), the minus sign in front of the first term of the right-hand side of the expenditure function should be deleted. That is, it should be

$$e(p,u) = p_2 u\exp(bp_1/p_2) - (1/b)(ap_1 + ap_2/b + p_2 c).$$

Finally, in (c), the minus sign in front of the first term of the right-hand side of the Hicksian function should be deleted. That is, it should be

$$h(p,u) = ub\exp(bp_1/p_2) - a/b.$$

(a) Use

$$\partial v(p,w)/\partial p_1 = -\ p_2^{-1}(ap_1/p_2 + bw/p_2 + c)\exp(-\ bp_1/p_2),$$
$$\partial v(p,w)/\partial w = p_2^{-1}\exp(-\ bp_1/p_2),$$

and apply Roy's formula.

(b) According to (3.E.1), we can obtain the expenditure function by solving

$$u = (e(p,u)/p_2 + b^{-1}(ap_1/p_2 + a/b + c))\exp(-\ bp_1/p_2).$$

(c) Apply Proposition 3.G.1 to obtain the given Hicksian demand function for the first good.

3.G.18 We prove the assertion by contradiction. Suppose that there exist $\ell \in \{1,\ldots,L\}$ and $k \in \{1,\ldots,L\}$ such that there is no chain of substitutes connecting ℓ and k. Define $J = \{\ell\} \cup \{j \in \{1,\ldots,L\}$: there is a chain of substitutes connecting ℓ and j$\}$. Since $\ell \in J$ and $k \notin J$, both J and its complement $\{1,\ldots,L\}\setminus J$ are nonempty. Moreover, for any $j \in J$ and any $j' \notin J$, $\partial h_j(p,u)/\partial p_{j'} < 0$, because, otherwise, $j' \in J$.

Let $u: \mathbb{R}^L_+ \to \mathbb{R}$ be the underlying utility function. Following the hint, as in Exercise 3.G.5, define $\tilde{u}: \mathbb{R}^2_+ \to \mathbb{R}$ by

$$\tilde{u}(y_1,y_2) = \text{Max}\{u(x): x \in \mathbb{R}^L_+, \textstyle\sum_{j\in J}p_jx_j \le y_1, \sum_{j\notin J}p_jx_j \le y_2\}.$$

Let $\tilde{h}: \mathbb{R}^2_{++} \times \tilde{u}(\mathbb{R}^2_+) \to \mathbb{R}^2_+$ be the Hicksian demand function derived from $\tilde{u}(\cdot)$. That is, $\tilde{h}(\alpha_1,\alpha_2,u) \in \mathbb{R}^2_+$ is the solution to

$$\text{Min}_{(y_1,y_2)}\ \alpha_1y_1 + \alpha_2y_2$$
$$\text{s.t.}\qquad \tilde{u}(y_1,y_2) \ge u.$$

Define $p(\alpha_1,\alpha_2) >> 0$ by $p_j(\alpha_1,\alpha_2) = \alpha_1p_j$ if $j \in J$ and $p_j(\alpha_1,\alpha_2) = \alpha_2p_j$ if $j \notin J$. We shall prove that

$$\tilde{h}(\alpha_1,\alpha_2,u) = (\textstyle\sum_{j\in J}p_jh_j(p(\alpha_1,\alpha_2),u), \sum_{j\notin J}p_jh_j(p(\alpha_1,\alpha_2),u)).$$

Write $x^* = h(p(\alpha_1,\alpha_2),u)$. Then $\tilde{u}(\sum_{j\in J}p_jx^*_j,\sum_{j\notin J}p_jx^*_j) \ge u(x^*) = u$. Hence the constraint of the cost minimization problem is satisfied. Suppose that $(y_1,y_2) \in \mathbb{R}^2_+$ and $\tilde{u}(y_1,y_2) \ge u$, then (assuming strong monotonicity) there exists $x \in \mathbb{R}^L_+$ such that $\sum_{j\in J}p_jx_j = y_1$, $\sum_{j\notin J}p_jx_j = y_2$, and $u(x) = \tilde{u}(y_1,y_2)$. Thus $u(x) \ge u$ and hence, by the cost minimization of x^*, $p(\alpha_1,\alpha_2)\cdot x \ge p(\alpha_1,\alpha_2)\cdot x^*$. This is equivalent to saying that

$$\alpha_1y_1 + \alpha_2y_2 \ge \alpha_1(\textstyle\sum_{j\in J}p_jx^*_j) + \alpha_2(\sum_{j\notin J}p_jx^*_j).$$

Thus $\tilde{h}(\alpha_1,\alpha_2,u) = (\sum_{j\in J}p_jx^*_j, \sum_{j\notin J}p_jx^*_j)$.

By this equality and the chain rule (M.A.1),

$$\partial\tilde{h}_1(\alpha_1,\alpha_2,u)/\partial\alpha_2 = \textstyle\sum_{j\in J}p_j(\sum_{k\notin J}(\partial h_j(p(\alpha_1,\alpha_2),u)/\partial p_k)p_k)$$
$$= \textstyle\sum_{j\in J}\sum_{k\notin J}p_jp_k(\partial h_j(p(\alpha_1,\alpha_2),u)/\partial p_k).$$

We now derive a contradiction from this equality evaluated at $(\alpha_1,\alpha_2) = (1,1)$. On the one hand, since $\partial h_j(p,u)/\partial p_k < 0$ for every $j \in J$ and every $k \notin J$, we must have $\partial\tilde{h}_1(1,1,u)/\partial\alpha_2 < 0$. On the other hand, note that $\tilde{h}(\cdot)$ is the

Hicksian demand function of $\tilde{u}(\cdot)$ for two (composite) goods. Since $\partial\tilde{h}_1(\alpha_1,\alpha_2,u)/\partial\alpha_1 \leq 0$ by the negative semidefiniteness, and

$$(\partial\tilde{h}_1(\alpha_1,\alpha_2,u)/\partial\alpha_1)p_1 + (\partial\tilde{h}_1(\alpha_1,\alpha_2,u)/\partial\alpha_2)p_2 = 0,$$

we must have $\partial\tilde{h}_1(1,1,u)/\partial\alpha_2 \geq 0$. We have thus obtained a contradiction.

3.H.1 By Proposition 3.H.1, $e(p,u) = \text{Min}\{p\cdot x\text{: } x \in V_u\}$. Thus, to complete the proof, it is sufficient to show that $V_u = \{x\text{: } u(x) \geq u\}$. That is, $x \in V_u$ if and only if $\text{Sup}\ \{t\text{: } x \in V_t\} \geq u$.

Clearly, if $x \in V_t$ then $\text{Sup}\ \{t\text{: } x \in V_t\} \geq u$.

Assume that $\text{Sup}\ \{t\text{: } x \in V_t\} \geq u$. Define $u^* = \text{Sup}\ \{t\text{: } x \in V_t\}$. If $u^* > u$, then there exists $t \in (u,u^*]$ such that $x \in V_t$. Since $e(\cdot)$ is increasing in utility levels, $V_u \supset V_t$ and hence $x \in V_u$. If $u^* = u$, then, for every $n \in \mathbb{N}$, there exists $u_n \in (u - 1/n, u)$ such that $x \in V_{u_n}$, that is, $p\cdot x \geq e(p,u_n)$ for all p. Let $n \to \infty$, then $u_n \to u$ and, by continuity of $e(p,u)$, $p\cdot x \geq e(p,u)$ for all p. Thus $x \in V_u$.

3.H.2 We show the contrapositive of the assertion. If a preference is not convex, then there exists at least one nonconvex upper contour set. Let $u \in \mathbb{R}$ be its corresponding utility level. We can choose a price vector p so that $h(p,u)$ consists of more than one elements, as the following figure shows. According to Proposition 3.F.1, $e(\cdot)$ is not differential at (p,u).

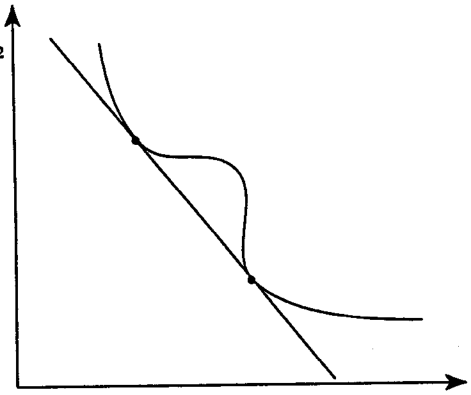


Figure 3.H.2

3.H.3 By (3.E.1), for each p, take the inverse of $e(p,u)$ with respect to u.

3.H.4 The following method is analogous to that of "Recovering the Expenditure Function from Demand" for L = 2.

Pick an arbitrary consumption vector x^0 and assign a utility value u^0 to x^0. We will now recover the indifference curve $\{x: u(x) = u^0\}$ going through x^0. Assuming strong mono tonicity, this is equivalent to finding a function $\xi(\cdot,u^0)$: $(0, \infty) \to (0, \infty)$ such that $u(x_1,\xi(x_1)) = u^0$ for every $x_1 > 0$.

Differentiate

both sides of $u(\bar{x}_1,\xi(\bar{x}_1)) = u^0$ with respect to $\bar{x}_1$, then we obtain

$$\partial u(\bar{x}_1,\xi(\bar{x}_1))/\partial x_1 + (\partial u(\bar{x}_1,\xi(\bar{x}_1))/\partial x_2)\xi'(\bar{x}_1) = 0.$$

Hence

$$\xi'(\bar{x}_1) = -\frac{\partial u(\bar{x}_1,\xi(\bar{x}_1))/\partial x_1}{\partial u(\bar{x}_1,\xi(\bar{x}_1))/\partial x_2}.$$

Since $\dfrac{\partial u(\bar{x}_1,\xi(\bar{x}_1))/\partial x_1}{\partial u(\bar{x}_1,\xi(\bar{x}_1))/\partial x_2} = \dfrac{g_1(\bar{x}_1,\xi(\bar{x}_1))}{g_2(\bar{x}_1,\xi(\bar{x}_1))}$, we have $\xi'(\bar{x}_1) = -\dfrac{g_1(\bar{x}_1,\xi(\bar{x}_1))}{g_2(\bar{x}_1,\xi(\bar{x}_1))}$, or, by replacing $\bar{x}_1$ by x_1, we obtain

$$\xi'(x_1) = -\frac{g_1(x_1,\xi(x_1))}{g_2(x_1,\xi(x_1))}.$$

By solving this differential equation, we obtain the indifference curve going through x^0.

3.H.5 By (3.E.1), we can recover the expenditure function by simply inverting the indirect utility function.

To recover the direct utility function, define u: $\mathbb{R}^L_+ \to \mathbb{R}$ by $u(x) = \text{Min}\{v(p,w): p\cdot x \le w\}$. We shall prove that $u(x)$ is the direct utility function that generates $v(p,w)$. So let $x^*(p,w)$ be the demand function and $v^*(p,w)$ be the indirect utility function generated by $u(x)$. It is sufficient to show that $v^*(p,w) = v(p,w)$ for all $p >> 0$ and $w \ge 0$.

Let $p >> 0$ and $w \ge 0$, then $p\cdot x^*(p,w) = w$ and hence $v^*(p,w) = u(x^*(p,w)) \le v(p,w)$. It thus remains to show that $v^*(p,w) \ge v(p,w)$. Define

$$x = -\frac{1}{\nabla_w v(p,w)} \nabla_p v(p,w).$$

Then $x \in \mathbb{R}^L_+$ by the monotonicity. Since $\nabla_p v(p,w)\cdot p + \nabla_w v(p,w)w = 0$ by the homogeneity, $p\cdot x = w$. It is thus sufficient to show that $u(x) \ge v(p,w)$. So let $p' >> 0$ and $w' \ge 0$ satisfy $p'\cdot x = w'$. Then $(p' - p)\cdot x = w' - w$, or, by the definition of x, $\nabla_p v(p,w)\cdot(p' - p) + \nabla_w v(p,w)(w' - w) = 0$. Hence, by the quasiconvexity of $v(p,w)$, $v(p',w') \ge v(p,w)$. Thus $u(x) \ge v(p,w)$.

3.H.6 Let's first prove that the symmetry condition on $S(p,w)$ is satisfied. By equation (3.H.2), $\partial e(p,u)/\partial p_\ell = \alpha_\ell e(p,u)/p_\ell$ for every ℓ. By differentiating both sides with respect to p_k, we obtain

$$\partial^2 e(p,u)/\partial p_\ell \partial p_k = (\alpha_\ell/p_\ell)(\partial e(p,u)/\partial p_k) = (\alpha_\ell \alpha_k/p_\ell p_k)e(p,u).$$

On the other hand, by differentiating both sides of $\partial e(p,u)/\partial p_k = \alpha_k e(p,u)/p_k$

with respect to p_ℓ, we obtain

$$\partial^2 e(p,u)/\partial p_k \partial p_\ell = (\alpha_k/p_k)(\partial e(p,u)/\partial p_\ell) = (\alpha_\ell \alpha_k / p_\ell p_k) e(p,u).$$

Hence the symmetry condition is satisfied.

We can thus apply the iterative method explained in the small-type discussion at the end of Section 3.H to derive the expenditure function. First, we shall prove by induction that, for every ℓ, there exists a function $f_\ell(p_{\ell+1},\ldots,p_L,u)$ such that

$$\ln e(p,u) = \textstyle\sum_{k\le\ell}\alpha_k \ln p_k + f_\ell(p_{\ell+1},\ldots,p_L,u).$$

Suppose first that $\ell = 1$. Since $(\partial e(p,u)/\partial p_1)/e(p,u) = \alpha_1/p_1$. Hence, by integrating both sides with respect to p_1, we obtain

$$\ln e(p,u) = \alpha_1 \ln p_1 + f_1(p_2,\ldots,p_L,u).$$

Thus the equality is verified for $\ell = 1$. Suppose next that $\ell > 1$ and the equality holds for $\ell - 1$. By differentiating both sides of

$$\ln e(p,u) = \textstyle\sum_{k\le\ell-1}\alpha_k \ln p_k + f_{\ell-1}(p_\ell,\ldots,p_L,u)$$

with respect to p_ℓ, we obtain

$$\partial e(p,u)/\partial p_\ell = \partial f_{\ell-1}(p_\ell,\ldots,p_L,u)/\partial p_\ell.$$

Since $\partial e(p,u)/\partial p_\ell = \alpha_\ell/p_\ell$, this is equivalent to

$$\alpha_\ell/p_\ell = \partial f_{\ell-1}(p_\ell,\ldots,p_L,u)/\partial p_\ell.$$

Hence, by integrating both sides with respect to p_ℓ, we know that there exists $f_\ell(p_{\ell+1},\ldots,p_L,u)$ such that

$$\alpha_\ell \ln p_\ell = f_{\ell-1}(p_\ell,\ldots,p_L,u) - f_\ell(p_{\ell+1},\ldots,p_L,u)$$

By plugging this into

$$\ln e(p,u) = \textstyle\sum_{k\le\ell-1}\alpha_k \ln p_k + f_{\ell-1}(p_\ell,\ldots,p_L,u),$$

we obtain

$$\ln e(p,u) = \textstyle\sum_{k\le\ell}\alpha_k \ln p_k + f_\ell(p_{\ell+1},\ldots,p_L,u).$$

If $\ell = L$, then this equality becomes $\ln e(p,u) = \sum_{k\le L}\alpha_k \ln p_k + f_L(u)$. Or, equivalently, $e(p,u) = (\Pi_\ell p_\ell^{\alpha_\ell})\exp f_L(u)$.

In what follows, for every increasing function $f_L(u)$, we shall find the utility function that generates the expenditure function $e(p,u) = (\Pi_\ell p_\ell^{\alpha_\ell})\exp f_L(u)$. To start, consider the utility function $u^*(x) = \Pi_\ell x_\ell^{\alpha_\ell}$, which appeared in Example 3.E.1 for the case of L = 2. Denote its expenditure function by $e^*(p,u)$, then

$$e^*(p,u^*) = (\Pi_\ell \alpha_\ell^{-\alpha_\ell})(\Pi_\ell p_\ell^{\alpha_\ell})u^*.$$

(We considered a similar expenditure function in Exercise 3.G.16. Note that, these similarities incidentally show that, for every increasing function $f_L(u)$, the expenditure function $e(p,u) = (\Pi_\ell p_\ell^{\alpha_\ell})\exp f_L(u)$ has all the properties of expenditure functions in Proposition 3.E.2, because it corresponds one of the monotone transformations of $u^*(x) = \Pi_\ell x_\ell^{\alpha_\ell}$.) Let $g(u^*)$ be an monotone transformation and denote by $e_g(p,u)$ the expenditure function of the utility function $(g \circ u^*)(x)$. Then

$$e_g(p,u) = e^*(p,g^{-1}(u)) = (\Pi_\ell \alpha_\ell^{-\alpha_\ell})(\Pi_\ell p_\ell^{\alpha_\ell})g^{-1}(u).$$

By comparing this with $e(p,u) = (\Pi_\ell p_\ell^{\alpha_\ell})\exp f_L(u)$, we know that $(g \circ u^*)(x) = u(x)$ if

$$(\Pi_\ell \alpha_\ell^{-\alpha_\ell})(\Pi_\ell p_\ell^{\alpha_\ell})g^{-1}(u) = (\Pi_\ell p_\ell^{\alpha_\ell})\exp f_L(u).$$

This equality is equivalent to $g^{-1}(u) = (\Pi_\ell \alpha_\ell^{\alpha_\ell})\exp f_L(u)$. Letting $u^* = g^{-1}(u)$ and solving this with respect to u^*, we obtain

$$g(u^*) = f_L^{-1}(\ln u^* - \textstyle\sum_\ell \alpha_\ell \ln \alpha_\ell).$$

Thus

$$u(x) = g(u^*(x)) = f_L^{-1}(\ln u^*(x) - \textstyle\sum_\ell \alpha_\ell \ln \alpha_\ell) = f_L^{-1}(\textstyle\sum_\ell \alpha_\ell(\ln x_\ell - \ln \alpha_\ell)).$$

Of course, two possible utility functions are $u(x) = \Pi_\ell x_\ell^{\alpha_\ell}$ (corresponding to $f_L(u) = \ln u - \sum_\ell \alpha_\ell \ln \alpha_\ell$) and $u(x) = \sum_\ell \alpha_\ell \ln x_\ell$ (corresponding to $f_L(u) = u -$

$\sum_\ell \alpha_\ell \ln\alpha_\ell)$.

3.H.7 **(a)** Let $\bar{p} = (1,\ldots,1)$. Since $x(\bar{p},L) = (1,\ldots,1)$ and $u(1,\ldots,1) = 1$, according to Propositions 3.E.1 and 3.G.1, $D_p e(\bar{p},1) = h(\bar{p},1) = (1,\ldots,1)$. Hence $e(\bar{p},1) = L$.

On the other hand, $S(q,w) = D_p^2 e(q,w) = 0$ for every $q \gg 0$ by Exercise 2.F.17(d). Hence

$$e(p,u) - e(q,u) = D_p e(q,u)(p - q).$$

for every $q \gg 0$ and $p \gg 0$. Now take $q = \bar{p}$, then $e(p,1) - L = \bar{p}\cdot(p - \bar{p})$. Thus $e(p,1) = \sum_{\ell=1}^{L} p_\ell$.

(b) The upper contour set is equal to

$$\{x \in \mathbb{R}_+^L: p\cdot x \geq \textstyle\sum_{\ell=1}^{L} p_\ell \text{ for every } p \gg 0\} = \{x \in \mathbb{R}_+^L: x \geq (1,\ldots,1)\}.$$

3.I.1 By the same method as deriving equation (3.I.3), we obtain

$$\begin{aligned}
&EV(p^0,p^1,w) = e(p^0,u^1) - e(p^1,u^1)\\
&= e(p^0,u^1) - e(p_1^1,p_2^0,p_3^0,\ldots,p_L^0,u^1) + e(p_1^1,p_2^0,p_3^0,\ldots,p_L^0,u^1) - e(p^1,u^1)\\
&= \int_{p_1^1}^{p_1^0} h(p_1,p_2^0,p_3^0,\ldots,p_L^0,u^1)\,dp_1 + \int_{p_2^1}^{p_2^0} h(p_1^1,p_2,p_3^0,\ldots,p_L^0,u^1)\,dp_2;\\
&CV(p^0,p^1,w) = e(p^0,u^0) - e(p^1,u^0)\\
&= e(p^0,u^0) - e(p_1^1,p_2^0,p_3^0,\ldots,p_L^0,u^0) + e(p_1^1,p_2^0,p_3^0,\ldots,p_L^0,u^0) - e(p^1,u^0)\\
&= \int_{p_1^1}^{p_1^0} h(p_1,p_2^0,p_3^0,\ldots,p_L^0,u^0)\,dp_1 + \int_{p_2^1}^{p_2^0} h(p_1^1,p_2,p_3^0,\ldots,p_L^0,u^0)\,dp_2.
\end{aligned}$$

If there is no wealth effect for either good, then, by the first relation of (3.E.4),

$$h(p_1,p_2^0,p_3^0,\ldots,p_L^0,u^1) = h(p_1,p_2^0,p_3^0,\ldots,p_L^0,u^0) \text{ for every } p_1 > 0,$$
$$h(p_1^1,p_2,p_3^0,\ldots,p_L^0,u^1) = h(p_1^1,p_2,p_3^0,\ldots,p_L^0,u^0) \text{ for every } p_2 > 0.$$

Thus $EV(p^0,p^1,w) = CV(p^0,p^1,w)$.

3.I.2 Denote the deadweight loss given in equation (3.I.5) by $DW_1(t)$ and that in equation (3.I.6) by $DW_0(t)$. Then

$$DW_1'(t) = h(p_1^0 + t,\ \bar{p}_{-1},\ u^1) - (h(p_1^0 + t,\ \bar{p}_{-1},\ u^1) + t\cdot\partial h(p_1^0 + t,\ \bar{p}_{-1},\ u^1)/\partial p_1)$$
$$= -\ t\cdot\partial h(p_1^0 + t,\ \bar{p}_{-1},\ u^1)/\partial p_1.$$

Thus $DW_1'(0) = 0$ and, if $\partial h(p_1,\bar{p}_{-1},u^1)/\partial p_1 > 0$ for every $p_1 > 0$, then $DW_1'(t) > 0$ for every $t > 0$. It can be similarly shown that $DW_0'(0) = 0$ and, if $\partial h(p_1,\bar{p}_{-1},u^0)/\partial p_1 > 0$ for every $p_1 > 0$, then $DW_0'(t) > 0$ for every $t > 0$.

A possible interpretation of this result is that the first-order derivatives of the deadweight loss at $t = 0$ may be a bit misleading approximation. In fact, their being zero means that, approximately, there is no deadweight loss. On the other hand, since those derivatives are positive at every $t > 0$, $DW_1(t) = \int_0^t DW_1'(\tau)\,d\tau > 0$ and $DW_0(t) = \int_0^t DW_0'(\tau)\,d\tau > 0$. Hence the deadweight losses are in fact positive.

3.I.3 Write $u^0 = v(p^0,w)$ and $u^1 = v(p^1,w)$, then $u^0 < u^1$ because $p^0 \geq p^1$ and $p^0 \neq p^1$. Thus

$$e(p_\ell,p^0_{-\ell},u^0) < h_\ell(p_\ell,p^0_{-\ell},u^1)$$

for every $p_\ell > 0$. Since good ℓ is inferior,

$$x_\ell(p_\ell,p^0_{-\ell},e(p_\ell,p^0_{-\ell},u^0)) > x_\ell(p_\ell,p^0_{-\ell},e(p_\ell,p^0_{-\ell},u^1)).$$

By the first relation of (3.E.4), this is equivalent to

$$h_\ell(p_\ell,p^0_{-\ell},u^0) > h_\ell(p_\ell,p^0_{-\ell},u^1).$$

Hence, by (3.I.3), (3.I,4), and $p^0_\ell < p^1_\ell$, we have $CV(p^0,p^1,w) > EV(p^0,p^1,w)$.

3.I.4 We shall give two examples, both of which have two commodities. The first one is simpler, while the second one is more illustrative.

In the first example, we consider a preference with "L-shaped" indifference curves such that the vectors (1,1), (4,2), and (5,3) are kinks of indifference curves. Let $u(1,1) = 1$. Note that if one of the two prices is equal to zero, then the demand is not a singleton. We thus need to consider a demand correspondence $x(p,w)$. But this does not essentially change our argument because we are working on expenditure functions, which is single-valued by its definition.

Let $p^0 = (1,1)$, $p^1 = (1/2,0)$, $p^2 = (0,2/3)$, and $w = 2$. Then $x(p^0,w) \ni (1,1)$, $x(p^1,w) \ni (4,2)$, $x(p^2,w) \ni (5,3)$, and $v(p^2,w) > v(p^1,w)$. But $e(p^1,1) = 1/2$ and $e(p^2,1) = 2/3$. Thus

$$CV(p^0,p^1,w) = 2 - 1/2 = 3/2,$$

$$CV(p^0,p^2,w) = 2 - 2/3 = 4/3.$$

Hence $CV(p^0,p^1,w) > CV(p^0,p^2,w)$.

It is worthwhile to remark that, although the given preference is neither smooth, strongly monotone, nor strictly convex, it can be approximated by such one.

In the second example, we consider a utility function $u(x)$ which is quasilinear with respect to the first commodity. Let $v(p,w)$ be the corresponding indirect utility function. Starting from $p^0 = (1,1)$ and $w > 0$, we consider two other price vectors $p^1 = (p_1^1,1)$ and $p^2 = (1,p_2^2)$ such that $0 < p_1^1 < 1$, $0 < p_2^2 < 1$, and $v(p^1,w) = v(p^2,w)$. Write $u^0 = v(p^0,w)$ and $u^1 = v(p^1,w) = v(p^2,w)$. Then $EV(p^0,p^1,w) = EV(p^0,p^2,w)$.

We shall now show that $CV(p^0,p^1,w) < CV(p^0,p^2,w)$. By $p_1^1 < 1$, $CV(p^0,p^1,w) < EV(p^0,p^1,w)$. Also, by the quasilinearity, $CV(p^0,p^2,w) = EV(p^0,p^2,w)$ (Exercise 3.I.5). Hence $CV(p^0,p^1,w) < CV(p^0,p^2,w)$.

It is worthwhile to remark that, although $EV(p^0,p^1,w) = EV(p^0,p^2,w)$, we can obtain the strict reverse inequality $EV(p^0,p^1,w) > EV(p^0,p^2,w)$, while preserving $CV(p^0,p^1,w) < CV(p^0,p^2,w)$, by decreasing p_2^2 only slightly.

3.I.5 According to Exercise 3.E.7, we can write the expenditure function $e(p,u) = \tilde{e}(p_2,\ldots,p_L) + u$ for $p_1 = 1$. Hence

$$\begin{aligned}
EV(p^0,p^1,w) &= e(p^0,u^1) - e(p^0,u^0) \\
&= (\tilde{e}(p_2^0,\ldots,p_L^0) + u^1) - (\tilde{e}(p_2^0,\ldots,p_L^0) + u^0) \\
&= u^1 - u^0. \\
CV(p^0,p^1,w) &= e(p^1,u^1) - e(p^1,u^0) \\
&= (\tilde{e}(p_2^1,\ldots,p_L^1) + u^1) - (\tilde{e}(p_2^1,\ldots,p_L^1) + u^0) \\
&= u^1 - u^0.
\end{aligned}$$

Hence $EV(p^0,p^1,w) = CV(p^0,p^1,w)$.

3.I.6 Let $u_i^0 = v_i(p^0,w_i)$. If $\sum_i CV_i(p^0,p^1,w_i) \geq 0$, then $\sum_i w_i \geq \sum_i e_i(p^1,u_i^0)$. So define $w_i' = e_i(p^1,u_i^0)$, then $\sum_i w_i' \leq \sum_i w_i$ and $v_i(p^1,w_i') = u_i^0 = v_i(p^0,w_i)$.

3.I.7 **(a)** By applying Walras' law and the homogeneity of degree zero, we can obtain the demand functions for all three good defined over the whole domain $\{(p,w) \in \mathbb{R}^3 \times \mathbb{R}: p \gg 0\}$. Thus we can obtain the whole 3×3 Slutsky matrix as well from the demand function. The 2×2 submatrix of the Slutsky matrix that is obtained by deleting the last row and the last column is equal to $(1/p_3)\begin{bmatrix} b & c \\ e & g \end{bmatrix}$. By the homogeneity and Walras' law, the 3×3 Slutsky matrix is symmetric if and only if this 2×2 matrix is symmetric. Moreover, just as in the proof of Theorem M.D.4(iii), we can show that the 3×3 Slutsky matrix is negative semidefinite (on T_p, and hence on the whole $\mathbb{R}^3$) if and only if the

2×2 matrix is negative semidefinite. Hence, utility maximization implies that $c = e$, $b \le 0$, $g \le 0$, and $bg - c^2 \ge 0$.

(b) First, we verify that the corresponding Hicksian demand functions for the first two commodities are independent of utility levels and, as functions of the prices of the first two commodities alone, they are equal to the given Walrasian demand functions. Let p be any price vector and u, u' be any two utility levels. By (3.E.4), $h_\ell(p,u) = x_\ell(p,e(p,u))$ and $h_\ell(p,u') = x_\ell(p,e(p,u'))$ for $\ell = 1,2$. Since the $x_\ell(\cdot)$ do not depend on wealth, $x_\ell(p,e(p,u)) = x_\ell(p,e(p,u'))$. Hence $h_\ell(p,u) = h_\ell(p,u')$. Thus the $h_\ell(p,u)$ do not depend on utility level and they are the same as the $x_\ell(p,w)$.

If the prices change following the path $(1,1) \to (2,1) \to (2,2)$, then the equivalent variation is

$$\begin{aligned} &\int_1^2 h^1(p^1,1,u)dp^1 + \int_1^2 h^2(2,p^2,u)dp^2 \\ &= \int_1^2 x^1(p^1,1,w)dp^1 + \int_1^2 x^2(2,p^2,w)dp^2 \\ &= (a + (3/2)b + c) + (d + 2e + (3/2)g). \end{aligned}$$

If the prices change following the path $(1,1) \to (1,2) \to (2,2)$, then the equivalent variation is

$$\begin{aligned} &\int_1^2 h^2(1,p^2,u)dp^2 + \int_1^2 h^1(p^1,2,u)dp^1 \\ &= \int_1^2 x^2(1,p^2,w)dp^2 + \int_1^2 x^1(p^1,2,w)dp^1 \\ &= (d + e + (3/2)g) + (a + (3/2)b + 2c). \end{aligned}$$

These two equivalent variations are the same if and only if $c = e$.

(c) As we saw above,

$$\begin{aligned} EV_1 &= \int_1^2 x^1(p^1,1,w)dp^1 = a + (3/2)b + c, \\ EV_2 &= \int_1^2 x^2(1,p^2,w)dp^2 = d + e + (3/2)g = d + c + (3/2)g, \\ EV\ &= (a + (3/2)b + c) + (d + 2e + (3/2)g) \end{aligned}$$

$$= a + (3/2)b + 3c + d + (3/2)g.$$

Hence $EV - (EV_1 + EV_2) = c$.

The sum $EV_1 + EV_2$ does not contain the effect on equivalent variation due to the shift of the graph of the demand function for the second commodity when p^1 goes up to 2 (or equivalently, the shift of the graph of the demand function for the first commodity when p^2 goes up to 2). Graphically, letting $c = e > 0$, EV contains the shaded area below but $EV_1 + EV_2$ does not:

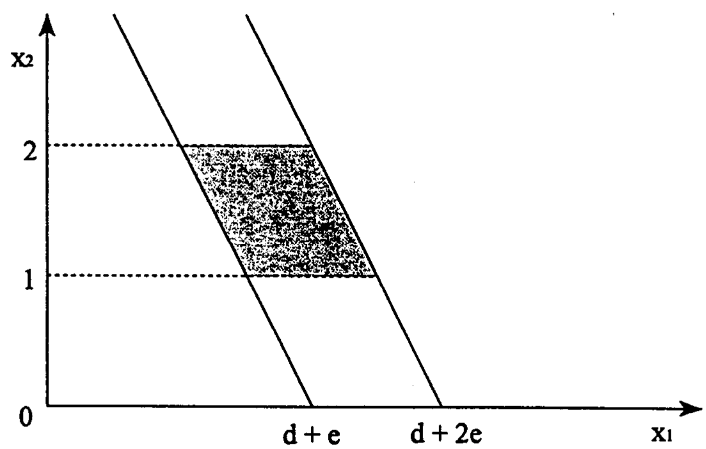


Figure 3.I.7(c)

(d) Since $x_1(2,1,w) = a + 2b + c$, the tax revenue from the first good is equal to this. Thus $DW_1 = (a + (3/2)b + c) - (a + 2b + c) = - b/2$.

Since $x_2(1,2,w) = d + e + 2g$, the tax revenue from the second good is equal to this. Thus $DW_2 = (d + e + (3/2)g) - (d + e + 2g) = - g/2$.

Since $x_1(2,2,w) = a + 2b + 2c$ and $x_2(2,2,w) = d + 2e + 2g$, the tax revenue from both commodities is

$$(a + 2b + 2c) + (d + 2e + 2g) = a + 2b + 4c + d + 2g.$$

Thus

$$DW = (a + (3/2)b + 3c + d + (3/2)g) - (a + 2b + 4c + d + 2g).$$

$= - b/2 - c - g/2.$

Hence $DW - (DW_1 + DW_2) = - c$.

(e) Our problem is

$$\text{Min}_{(t_1,t_2)}\ DW(t_1,t_2)$$

$$\text{s.t. } \textstyle\sum_{\ell=1}^{2} h_\ell(1 + t_1,\ 1 + t_2,\ u)t_\ell \geq R.$$

Here,

$$DW(t_1,t_2) = EV(t_1,t_2) - TR(t_1,t_2)$$
$$= e(1 + t_1,\ 1 + t_2,\ u) - e(1,1,u) - \textstyle\sum_{\ell=1}^{2} h_\ell(1 + t_1,\ 1 + t_2,\ u)t_\ell.$$

Set up the Lagrangean by $L(t_1,t_2,\lambda) = DW(t_1,t_2) + \lambda(R - TR(t_1,t_2))$. Then the first-order condition with respect to t_ℓ is $\partial DW(t_1,t_2)/\partial t_\ell - \lambda\partial TR(t_1,t_2)/\partial t_\ell = 0$. But,

$$\partial DW(t_1,t_2)/\partial t_\ell = \partial e(1 + t_1,\ 1 + t_2,\ u)/\partial t_\ell - h_\ell(1 + t_1,\ 1 + t_2,\ u)$$
$$- \textstyle\sum_{k=1}^{2}(\partial h_k(1 + t_1,\ 1 + t_2,\ u)/\partial t_\ell)t_k$$
$$= - \textstyle\sum_{k=1}^{2}(\partial h_k(1 + t_1,\ 1 + t_2,\ u)/\partial t_\ell)t_k$$

by $\partial e(1 + t_1,\ 1 + t_2,\ u)/\partial t_\ell = h_\ell(1 + t_1,\ 1 + t_2,\ u)$, and

$$\partial TR(t_1,t_2)/\partial t_\ell = h_\ell(1 + t_1,\ 1 + t_2,\ u) + \textstyle\sum_{k=1}^{2}(\partial h_k(1 + t_1,\ 1 + t_2,\ u)/\partial t_\ell)t_k.$$

Hence the first-order condition is written as

$$\textstyle\sum_{k=1}^{2}(\partial h_k(1 + t_1,\ 1 + t_2,\ u)/\partial t_\ell)t_k(1 + \lambda)$$
$$+ \lambda h_\ell(1 + t_1,\ 1 + t_2,\ u) = 0$$

for both $\ell = 1,2$. From this and $R = \sum_{\ell=1}^{2} h_\ell(1 + t_1,\ 1 + t_2,\ u)t_\ell$, we obtain

$$- \lambda = \frac{bt_1 + ct_2}{a + b(1 + 2t_1) + c(1 + 2t_2)} = \frac{ct_1 + gt_2}{a + c(1 + 2t_1) + g(1 + 2t_2)},$$

$$(a + b(1 + t_1) + c(1 + t_2))t_1 + (d + c(1 + t_1) + g(1 + t_2))t_2 = R.$$

3.I.8 **(a)** Quasilinear utility functions: $u(x_1,x_2,x_3) = \tilde{u}(x_1,x_2) + x_3$.

(b) As in Exercise 3.I.7(a), the symmetry implies

$$c_1 + d_1p_1 = b_2 + d_2p_2 \text{ for all } p_1 > 0 \text{ and } p_2 > 0.$$

Thus $c_1 = b_2$, $d_1 = d_2 = 0$. Then the negative semidefiniteness implies that

$$b_1 \le 0,\ c_2 \le 0,\ b_1c_2 - c_1^2 \le 0.$$

(c) Since the Walrasian demand functions and Hicksian demand functions are the same as we saw in Exercise 3.I.7(b), we can define

$$CV = \int_{p_1}^{p_1'} x_1(q,p_2,w)dq + \int_{p_2}^{p_2'} x_1(p_1',q,w)dq,$$

or, equivalently,

$$CV = \int_{p_2}^{p_2'} x_2(p_1,q,w)dq + \int_{p_1}^{p_1'} x_1(q,p_2',w)dq.$$

(d) By the same calculation as in Exercise 3.I.7(c), we obtain

$$EV_1 = 1/2,\ EV_2 = 1/2,\ EV_3 = 3/2.$$

In this case, $EV \ne EV_1 + EV_2$. In the general case in which the conditions in (b) hold,

$$EV_1 = a_1 + (3/2)b_1 + c_1,$$
$$EV_2 = a_2 + b_2 + (3/2)c_2,$$
$$EV_3 = a_1 + (3/2)b_1 + c_1 + a_2 + 2b_2 + (3/2)c_2.$$

Hence $EV_1 + EV_2 = EV_3$ if and only if $b_2 = c_1 = 0$. This condition is equivalent to saying that any change in the price of one good does not have any (cross) effect on the demand for the other.

3.I.9 Let $e_\ell \in \mathbb{R}^L$ be the vector of which the ℓ-th component is one and all the other components are zero. For each t, define $p(t) = p + te_\ell$. Then the after-rebate income $w(t)$ with tax t satisfies $w(t) = w + x_\ell(p(t),w(t))t$.

Hence $p \cdot x(p(t),w(t)) = (p(t) - te_\ell) \cdot x(p(t),w(t)) = w(t) - x_\ell(p(t),w(t))t = w$. Therefore $x(p(t),w(t))$ is at most as good as $x(p,w)$. In order to prove that $x(p(t),w(t))$ is strictly less preferred to $x(p,w)$, it is sufficient to prove that these two are different, because the demand function is assumed to be single-valued.

Now suppose that there exists a $t > 0$ such that $x(p(t),w(t)) = x(p,w)$. Let $u = v(p,w)$. Then we have $h(p(t),u) = h(p,u)$. In particular, $h_\ell(p(t),u) = h_\ell(p,u)$. Since the Hicksian demand function $s \to h_\ell(p(s),u)$ is nonincreasing, this equality implies that $h_\ell(p(s),u) = h_\ell(p,u)$ for every $s \in [0,t]$. But $d[h_\ell(p(s),u)]/ds = \partial h_\ell(p(s),u)/\partial p_\ell$. Evaluating at $s = 0$, we have $\partial h_\ell(p,u)/\partial p_\ell = s_{\ell\ell}(p,w) = 0$. This violates the assumption that $s_{\ell\ell}(p,w) < 0$. Hence $h(p(t),w(t)) \neq h(p,w)$ for every $t > 0$.

3.I.10 We consider an example of a consumer who face the choices over two goods and whose preference $\succsim$ and demand function $x(p_1,p_2,w)$ satisfy the following condition:

For every $p_1 \in [1,2]$, $x(p_1,1,2) = ((1 - \varepsilon)/p_1,\ 1 + \varepsilon)$;

for every $p_2 \in [1,2]$, $x(1,p_2,2) = (1 - \varepsilon,\ (1 + \varepsilon)/p_2)$;

$((1 - \varepsilon)/2,\ 1 + \varepsilon) \succ (1 - \varepsilon,\ (1 + \varepsilon)/2)$.

By using the figure below, you can convince yourself, perhaps with some application of the weak axiom, that there actually exists such a preference.

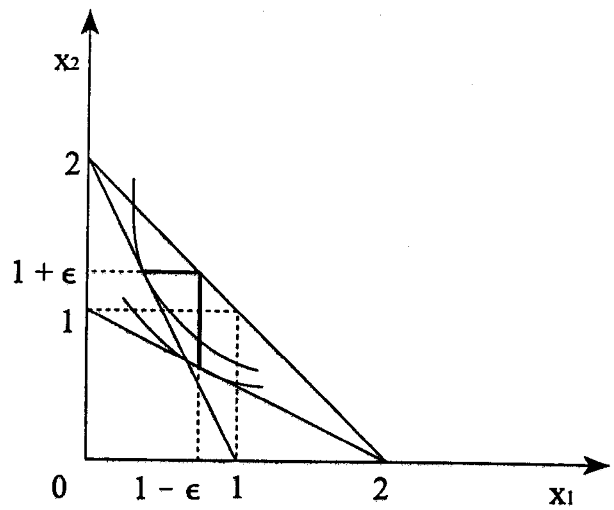


Figure 3.I.10

Define $p^0 = (2,1)$ and $p^1 = (1,2)$. Then

$$x(p^0,2) = ((1 - \varepsilon)/2,\ 1 + \varepsilon),$$

$$x(p^1,2) = (1 - \varepsilon,\ (1 + \varepsilon)/2).$$

Thus $x(p^0,2) \succ x(p^1,2)$. However, the area variation measure following the price-change path $p^0 \to (1,1) \to p^1$ is

$$\begin{aligned} AV(p^0,p^1,2) &= \textstyle\int_2^1 x_1(p_1,1,2)dp_1 + \int_1^2 x_2(1,p_2,2)dp_2 \\ &= -\textstyle\int_1^2 (1 - \varepsilon)/p_1\, dp_1 + \int_1^2 (1 + \varepsilon)/p_2\, dp_2 \\ &= -[(1 - \varepsilon)\ln p_1]_{p_1=1}^2 + [(1 + \varepsilon)\ln p_2]_{p_1=1}^2 \\ &= 2\varepsilon\ln 2 > 0. \end{aligned}$$

Hence the area variation measure ranks p^1 over p^0.

3.I.11 If $(p^1 - p^0)\cdot x^1 > 0$, then $w > p^0\cdot x^1$. The local non-satiation implies that x^0 is preferred to x^1. Hence the consumer must be worse off at (p^1,w).

As for the interpretation in term of the first-order approximation, since $e(p,u)$ is concave in p,

$$e(p^0,u^1) \le e(p^1,u^1) + \nabla e(p^1,u^1)\cdot(p^0 - p^1).$$

Since $\nabla e(p^1,u^1)\cdot(p^0 - p^1) < 0$, $e(p^0,u^1) < e(p^1,u^1) = w$. Thus $u^0 = v(p^0,w) > u^1$.

Finally, $(p^1 - p^0)\cdot x^1 > 0$ if and only if $w > p^0\cdot x^1$, which, in turn, is equivalent to $p^0\cdot(x^1 - x^0) < 0$. This test is depicted in the picture below:

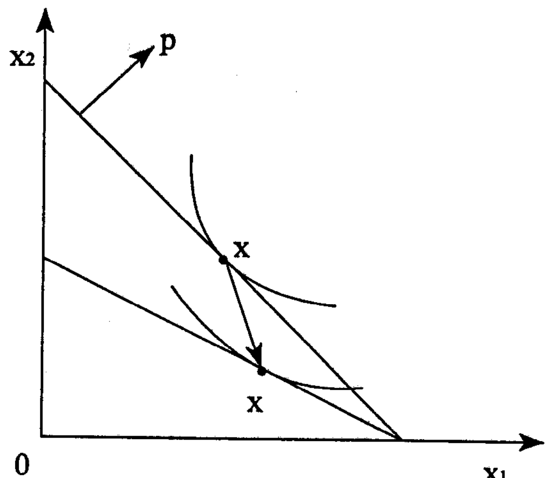


Figure 3.I.11

3.I.12 Let $u^0 = v(p^0,w^0)$ and $u^1 = v(p^1,w^1)$. Then we define

$$EV(p^0,w^0;p^1,w^1) = e(p^0,u^1) - e(p^0,u^0) = e(p^0,u^1) - w^0,$$

$$CV(p^0,w^0;p^1,w^1) = e(p^1,u^1) - e(p^1,u^0) = w^1 - e(p^1,u^0).$$

The "partial information" test can be extended as follows: If $p^1\cdot x^0 < w^1$, then the consumer is better off at (p^1,w^1). This can be proved in three ways. The first one is the same revealed-preference argument as in the proof of Proposition 3.I.1.

The second way is to use the indirect utility function. Since $v(p,w)$ is quasiconvex, if

$$(p^1 - p^0)\cdot\nabla_p v(p^0,w^0) + (w^1 - w^0)\partial v(p^0,w^0)/\partial w > 0,$$

then we can conclude that $v(p^1,w^1) > v(p^0,w^0)$. But, by Roy's identity, this sufficient condition is equal to

$$- (p^1 - p^0)\cdot(\partial v(p^0,w^0)/\partial w)x(p^0,w^0) + (w^1 - w^0)(\partial v(p^0,w^0)/\partial w)$$

$$= (\partial v(p^0,w^0)/\partial w)(- p^1\cdot x^0 + w^0 + w^1 - w^0)$$

$$= (\partial v(p^0,w^0)/\partial w)(w^1 - p^1\cdot x^0) > 0.$$

Hence, if $p^1\cdot x^0 < w^1$, then $v(p^1,w^1) > v(p^0,w^0)$.

The third way is to use the expenditure function. $v(p^1,w^1) > v(p^0,w^0)$ if and only if $e(p^1,v(p^1,w^1)) > e(p^1,v(p^0,w^0))$. But $e(p^1,v(p^1,w^1)) = w^1$ and $e(p^1,v(p^0,w^0)) \le p^1\cdot x^0$. Hence, if $p^1\cdot x^0 < w^1$, then we can conclude that $v(p^1,w^1) > v(p^0,w^0)$.

3.J.1 [First printing errata: The difficulty level should probably be B.] It follows immediately from the definition that if $x(p,w)$ satisfies the strong axiom, then it satisfied the weak axiom. Conversely, if $x(p,w)$ satisfies the weak axiom (in addition to the homogeneity of degree zero and Walras' law), then the Slutsky matrix is negative semidefinite and, by Exercise 2.F.11, symmetric. Hence $x(p,w)$ is integrable, implying that there exists a preference relation that generates $x(p,w)$. Thus $x(p,w)$ satisfies the strong axiom as well.

3.AA.1 If $(p,w) = (1,1,1)$, then $x(p,w) = (0,1)$. The locally cheaper condition is not satisfied since $B_{p,w} = \{x \in \mathbb{R}^L_+: x_1 + x_2 = 1\}$ and there is no y such that $p\cdot y < w$, as depicted in the following figure.

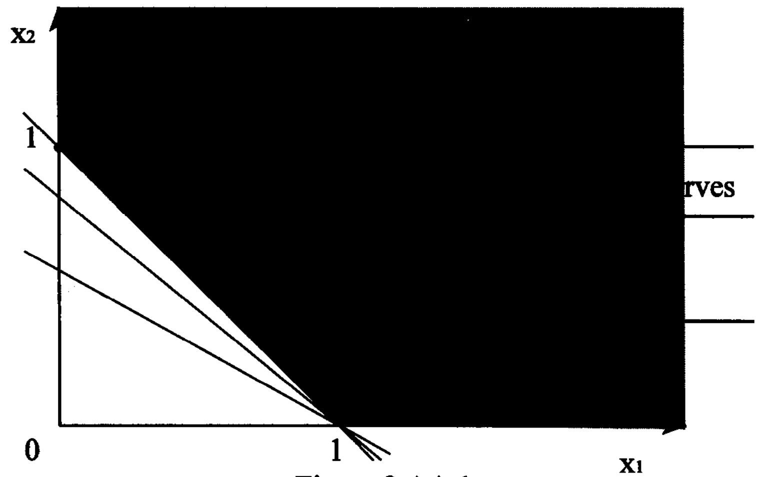


Figure 3.AA.1

To check that the demand function is not continuous at (1,1,1), consider the sequence $(p^n, w^n) = (1 - 1/n, 1, 1 - 1/n)$. Then $(p^n, w^n) \to (1,1,1)$ and $x(p^n, w^n) = (1,0)$, but $x(1,1,1) = (0,1)$. This discontinuous change in demands arises because the budget set B_{p^n,w^n} consists of (1,0) for every n, but $B_{(1,1),1} = \{x \in \mathbb{R}^2_+ : x_1 + x_2 = 1\}$, so that the commodity bundle (0,1) becomes available suddenly at p = (1,1).

3.AA.2 [First printing errata: The upper hemicontinuity of $h(p,u)$ cannot be guaranteed at $p \geq 0$, because the local boundedness condition in the definition of upper hemicontinuity need not be satisfied. Hence the clause in the bracket "even if we replace minimum by infimum and allow $p \geq 0$" should be understood as concerning only with $e(p,u)$.] We shall first prove that $h(p,u)$ is upper hemicontinuous. Let B be a compact subset of the domain of $h(p,u)$ (which is, in turn, a subset of $\{p \in \mathbb{R}^L : p \gg 0\} \times \mathbb{R}$). Then there exists a $(\bar{p},\bar{u}) \in B$ such that $\bar{u} \geq u$ for every $(p,u) \in B$. Let $\bar{x} \in h(\bar{p},\bar{u})$, then $u(\bar{x}) \geq \bar{u} \geq u$ for every $(p,u) \in B$. For each ℓ, define

$$\bar{y}_\ell = \text{Max}\{p\cdot\bar{x}/p_\ell \in \mathbb{R}_+ : (p,u) \in B\}$$

and $\bar{y} = (\bar{y}_1,\dots,\bar{y}_L) \in \mathbb{R}^L_+$. We now show that, for every $(p,u) \in B$ and $x \in h(p,u)$, we have $\bar{y} \geq x$. If fact, since $u(\bar{x}) \geq u$, $p\cdot\bar{x} \geq p\cdot x$. Since $p \gg 0$ and $x \in \mathbb{R}^L_+$, $p_\ell x_\ell \leq p\cdot x$. Thus $p\cdot\bar{x} \geq p_\ell x_\ell$. Divide both side by p_ℓ, then we obtain $p\cdot\bar{x}/p_\ell \geq x_\ell$ and hence $\bar{y}_\ell \geq x_\ell$. We have therefore established the local boundedness condition of upper hemicontinuity. Next, let $\{(p^n,u^n)\}_n$ be a sequence of pairs of price vectors and utility levels, converging to (p,u). Let $\{x^n\}_n$ be a sequence in $\mathbb{R}^L_+$, $x^n \in h(p^n,u^n)$ for every n, and $x^n \to x$. It is sufficient to prove that $x \in h(p,u)$. Since $u(x^n) \geq u^n$ and $u(x^n) \to u(x)$ by the continuity, we obtain $u(x) \geq u$. Hence x satisfies the constraint of the EMP at (p,u). To show that it is cost-minimizing, let $y \in \mathbb{R}^L_+$ and $u(y) \geq u$. If $u(y) > u$, then $u(y) > u^n$ for any sufficiently large n. Hence $p^n\cdot y \geq p^n\cdot x^n$ for such n. By taking the limit as $n \to \infty$, we obtain $p\cdot y \geq p\cdot x$. Suppose then that $u(y) = u$. By the local nonsatiation, there exists a sequence $\{y^n\}_n$ in $\mathbb{R}^L_+$ such that $u(y^n) > u(x)$ for every n and $y^n \to x$. Hence there exists a subsequence $\{(p^{k(n)},u^{k(n)})\}_n$ of $\{(p^n,u^n)\}_n$ such that $u(y^n) \geq u^{k(n)}$. Hence $p^{k(n)}\cdot y^n \geq p^{k(n)}\cdot x^{k(n)}$. By taking the limit as $n \to \infty$, we obtain $p\cdot y \geq p\cdot x$. Hence x is cost-minimizing.

We now turn to the continuity of $e(p,u)$. In fact, its continuity at every $p \gg 0$ can be derived immediately the continuity of $h(p,u)$, as the latter is well defined at every $p \gg 0$. Thus the essential part of the following proof is the case of nonnegative, but not strictly positive, price vectors. We shall establish the continuity with respect to p and that with respect to u separately.

Let $u \in \mathbb{R}$ be a utility level, $p \in \mathbb{R}^L_+$ be a price vector, and $\{p^n\}$ be a sequence of price vectors in $\mathbb{R}^L_+$ converging to p. We need to prove that

$e(p^n,u) \to e(p,u)$. As a preliminary result, let's first prove that if the sequence $\{e(p^n,u)\}$ in $\mathbb{R}_+$ converges, then it must do so to $e(p,u)$. Let w be the limit of $\{e(p^n,u)\}$. Let $x \in \mathbb{R}^L_+$ and $u(x) \geq u$. Then $p^n \cdot x \geq e(p^n,u)$ for every n. Taking the limit as $n \to \infty$, we obtain $p \cdot x \geq w$. Since this holds for every $x \in \mathbb{R}^L_+$ with $u(x) \geq u$, we have $e(p,u) \geq w$. To prove the reverse inequality $e(p,u) \leq w$, we use the concavity of $e(p,u)$ in $p \in \mathbb{R}^L_+$. Take a subsequence $\{p^{k(n)}\}$ of $\{p^n\}$ such that $(p^{k(n)}_\ell - p_\ell)(p^{k(m)}_\ell - p_\ell) \geq 0$ for all $\ell \in \{1,\ldots,L\}$ and positive integers n and m. That is, for each $\ell \in \{1,\ldots,L\}$, we require the sign of $p^{k(n)}_\ell - p_\ell$ along the subsequence to be constant (including zero). Such a subsequence does actually exist because each $p^n - p$ has one of at most 2^L sign patterns. Now, for each $\ell \in \{1,\ldots,L\}$, let $v_\ell = 1$ if $p^{k(n)}_\ell - p_\ell \geq 0$ for every n; and $v_\ell = -1$ if $p^{k(n)}_\ell - p_\ell \leq 0$ for every n. Then,

$$p^{k(n)}_\ell = p_\ell + |p^{k(n)}_\ell - p_\ell| v_\ell.$$

So let $z^n_\ell = |p^{k(n)}_\ell - p_\ell| \geq 0$ and define $v^\ell \in \mathbb{R}^L$ by letting $v^\ell_\ell = v_\ell$ and $v^\ell_k = 0$ for every $k \neq \ell$, then

$$p^{k(n)} = p + \textstyle\sum_\ell z^n_\ell v^\ell.$$

Now, define $z^n_0 = 1 - \sum^L_{\ell=1} z^n_\ell$, then

$$p^{k(n)} = z^n_0 p + \textstyle\sum_\ell z^n_\ell (p + v^\ell).$$

Since $p^{k(n)} \to p$, $z^n_\ell \to 0$ for every $\ell \in \{1,\ldots,L\}$. Thus $z^n_0 \to 1$. Hence, for every sufficiently large n, $z^n_0 > 0$ and $p^{k(n)}$ is a convex combination of z^n_0, z^n_1, ..., z^n_L. Therefore, by the concavity,

$$e(p^{k(n)},u) \leq z^n_0 e(p,u) + \textstyle\sum_\ell z^n_\ell e(p + v^\ell, u).$$

Since $e(p^n,u) \to w$, $e(p^{k(n)},u) \to w$. The right-hand side converges to $e(p,u)$. Therefore $w \leq e(p,u)$.

We have thus proved our preliminary fact that if the sequence $\{e(p^n,u)\}$ in $\mathbb{R}_+$ converges, then it must do so to $e(p,u)$. Let's now prove by contradiction that this implies that $e(p^n,u) \to e(p,u)$. So suppose not, then there are a $\delta > 0$ and a subsequence $\{p^{k(n)}\}$ of $\{p^n\}$ such that

$$|e(p^{k(n)},u) - e(p,u)| \geq \delta$$

for all n. Since the subsequence $\{e(p^{k(n)},u)\}$ is bounded, it has a further, convergent subsequence. On the one hand, the limit can never be $e(p,u)$, because $|e(p^{k(n)},u) - e(p,u)| \geq \delta$ for all n. On the other hand, our preliminary result implies that the limit must be $e(p,u)$. This is a contradiction. We must thus have $e(p^n,u) \to e(p,u)$.

Let's now turn to the continuity of $e(p,u)$ with respect to u. Let $p \in \mathbb{R}^L_+$ be a price vector, $u \in \mathbb{R}$ be a utility level, and $\{u^n\}$ be a sequence of utility levels in $\mathbb{R}$ converging to u. We need to prove that $e(p,u^n) \to e(p,u)$. Just as before, it is sufficient to prove that if the sequence $\{e(p,u^n)\}$ in $\mathbb{R}_+$ converges, then it must do so to $e(p,u)$. Let w be the limit of $\{e(p,u^n)\}$. Let $\varepsilon > 0$, $x \in \mathbb{R}^L_+$, $u(x) \geq u$, and $p \cdot x < e(p,u) + \varepsilon$. By the local nonsatiation, we can make $u(x) > u$ while preserving $p \cdot x < e(p,u) + \varepsilon$. Then there exists a positive integer N such that $u(x) > u^n$ for every $n > N$. Thus, for such n, $p \cdot x \geq e(p,u^n)$. Take the limit as $n \to \infty$, then $p \cdot x \geq w$. Thus $e(p,u) + \varepsilon > w$. Since this holds for every $\varepsilon > 0$, we must have $e(p,u) \geq w$. To show the reverse inequality $e(p,u) \leq w$, we can assume that $u^n \leq u$ for every n. (The reason is as follows: If there is a subsequence such that $u^n \leq u$ for every n in the subsequence, then we can apply this case to subsequence. If there is no such subsequence, then there is a subsequence such that $u^n \geq u$ for every n in the subsequence. Hence $e(p,u) \leq e(p,u^n)$ for such n. Taking the limit, we obtain $e(p,u) \leq w$.) Now let $\bar{x} \in \mathbb{R}^L_+$ and $u(\bar{x}) \geq u$. Define $B = \{x \in \mathbb{R}^L_+ : \bar{x} \geq x\}$, then B is compact. This and $u^n \leq u$ implies that the truncated EMP

$$\text{Min } p \cdot x \quad \text{s.t } u(x) \geq u^n$$

has a solution, denoted by $x^n \in B$. Then $x^n \in h(p,u^n)$, that is, x^n a solution to the original, untruncated EMP, because $p \in \mathbb{R}^L_+$. Since B is compact, there is a convergent subsequence $\{x^{k(n)}\}$ of $\{x^n\}$. Denote its limit by x. Since $u(\cdot)$ is continuous, $u(x) \geq u$ and hence $p \cdot x \geq e(p,u)$. Moreover, $p \cdot x^{k(n)} = e(p,u^{k(n)})$ and $p \cdot x^{k(n)} \to p \cdot x$. Thus $w = p \cdot x$ and hence $w \geq e(p,u)$.

Suppose that $u(x)$ is strictly quasiconcave, twice continuously differentiable and that $\nabla u(x) \neq 0$ for all x. Then we know that $h(p,u)$ is a function and the Lagrange multiplier λ of the EMP must be positive. The first-order condition for the EMP can be considered as a system of L + 1 equations and L + 1 unknowns:

$$p - \lambda \nabla u(x) = 0$$

$$u(x) - u \quad = 0$$

By the implicit function theorem (Theorem M.E.1), the solution $h(p,u)$ as a function of the parameters (p,u) of the system is differentiable if the Jacobian of this system has a nonzero determinant

$$\begin{vmatrix} -D^2u(x) & -p \\ \nabla u(x)^T & 0 \end{vmatrix} \neq 0$$

at (p,x) satisfying the above two equations. But, then, $p = \lambda \nabla u(x)$ and hence this condition is equivalent to

$$\begin{vmatrix} -D^2u(x) & -\nabla u(x) \\ \nabla u(x)^T & 0 \end{vmatrix} \neq 0,$$

that is,

$$\begin{vmatrix} D^2u(x) & \nabla u(x) \\ \nabla u(x)^T & 0 \end{vmatrix} \neq 0.$$

By Theorem M.D.3(i), this inequality holds if $D^2u(x)$ is negative definite on $\{y \in \mathbb{R}^L: \nabla u(x) \cdot y = 0\}$. This sufficient condition is a stronger differential version of quasiconcavity, as the latter is equivalent to the condition that $D^2u(x)$ is negative semidefinite on $\{y \in \mathbb{R}^L: \nabla u(x) \cdot y = 0\}$.

CHAPTER 4

4.B.1 By Roy's identity (Proposition 3.G.4) and $v_i(p,w_i) = a_i(p) + b(p)w_i$,

$$x_i(p,w_i) = -\frac{1}{\nabla_{w_i} v_i(p,w_i)} \nabla_p v_i(p,w_i) = -\frac{1}{b(p)} \nabla_p a_i(p) - \frac{w_i}{b(p)} \nabla_p b(p).$$

Thus $\nabla_{w_i} x_i(p,w_i) = -\frac{1}{b(p)} \nabla_p b(p)$ for all i. Since the right-hand side is identical for every i, the set of consumers exhibit parallel, straight expansion paths.

As for the second part, by (3.E.1),

$$e_i(p,u_i) = (u_i - a_i(p))/b(p).$$

Hence, by letting $c(p) = 1/b(p)$ and $d_i(p) = -a_i(p)/b(p)$, we obtain $e_i(p,u_i) = c(p)u_i + d_i(p)$.

4.B.2 **(a)** Let $p \in \mathbb{R}^L$ be a price vector and $w \geq 0$ be an aggregate wealth. Consider two consumers, i and j. Consider two wealth distributions $(w_1,\ldots,w_I)$ and $(w'_1,\ldots,w'_I)$ such that $w_i = w'_j = w \geq 0$, $w_k = 0$ for any $k \neq i$, and $w'_k = 0$ for any $k \neq j$. Since the preferences are homothetic, $x(p,0,s_k) = 0$ for every k. Thus the aggregate demand with $(w_1,\ldots,w_I)$ is $x(p,w,s_i)$ and the aggregate demand with $(w'_1,\ldots,w'_I)$ is $x(p,w,s_j)$. Since aggregate demand depends only on prices and aggregate wealth, we have $x(p,w,s_i) = x(p,w,s_j)$. Since p and w were arbitrarily chosen, this means that i and j have the same demand function. Hence they have the same preference. Since i and j were arbitrarily chosen, we conclude that all consumers have the same preference.

(b) By analogy to the Gorman form, consider the following form of indirect utility functions:

$$v_i(p,w_i,s_i) = a_i(p) + b(p)w_i + c(p)s_i.$$

Note that $b(p)$ and $c(p)$ do not depend on i. By this and Roy's identity (Proposition 3.G.4),

$$x(p,w_i,s_i) = -\frac{1}{\nabla_{w_i} v_i(p,w_i)} \nabla_p v_i(p,w_i)$$
$$= -\frac{1}{b(p)}\nabla a_i(p) - \frac{w_i}{b(p)}\nabla b(p) - \frac{s_i}{b(p)}\nabla c(p)$$

Thus

$$\sum_i x(p,w_i,s_i) = -\frac{1}{b(p)}\sum_i \nabla a_i(p) - \frac{\sum_i w_i}{b(p)}\nabla b(p) - \frac{\sum_i s_i}{b(p)}\nabla c(p).$$

Thus the aggregate demand depends only on $\sum_i w_i$ and $\sum_i s_i$ (and p).

4.C.1 By the definition of a directional partial derivative,

$$D_p x(p,w)dp = \lim_{\varepsilon\to 0}(1/\varepsilon)(x(p + \varepsilon dp,\ w) - x(p,w)).$$

Hence

$$dp\cdot D_p x(p,w)dp = dp\cdot(\lim_{\varepsilon\to 0}(1/\varepsilon)(x(p + \varepsilon dp,\ w) - x(p,w)))$$
$$= \lim_{\varepsilon\to 0}(1/\varepsilon)dp\cdot(x(p + \varepsilon dp,\ w) - x(p,w))$$

But the ULD property implies that $dp\cdot(x(p + \varepsilon dp,\ w) - x(p,w)) \geq 0$ for all $\varepsilon >$ 0. Hence, by taking the limit $\varepsilon \to 0$, we obtain $dp\cdot D_p x(p,w)dp \leq 0$. Thus $D_p x(p,w)$ is negative semidefinite.

We shall prove the converse by contradiction. Suppose that the Jacobian $D_p x(p,w)$ is negative definite for all (p,w) and that there exist $p \in \mathbb{R}^L$, $p' \in \mathbb{R}^L$, and $w_i \in \mathbb{R}$ such that $x_i(p,w_i) \neq x_i(p',w_i)$ and

$$(p' - p)\cdot(x_i(p',w_i) - x_i(p,w_i)) \geq 0.$$

Let $\bar{\lambda} > 1$ be sufficiently close to 1 that for every $\lambda \in [0,\bar{\lambda}]$, demand is well defined at $(1 - \lambda)p + \lambda p'$. (If demand is well defined at strictly positive price vectors, $\bar{\lambda}$ is determined so that $(1 - \lambda)p + \lambda p' \gg 0$ for every $\lambda \in [0,\bar{\lambda}]$.) Define $p(\lambda) = (1 - \lambda)p + \lambda p'$ and

$$w_i(\lambda) = (p' - p)\cdot(x_i(p(\lambda),w_i) - x_i(p,w_i)).$$

Then the function $w(\cdot)$ is differentiable, $w_i(0) = 0$, $w_i(1) \geq 0$, and

$$w_i'(\lambda) = (p' - p)\cdot D_p x_i(p(\lambda),w_i)(p' - p).$$

We consider two cases:

Case 1: $w_i(\lambda) \leq 0$ for every $\lambda \in [0,\bar{\lambda}]$.

Then $w_i(1) = 0$ and it is a maximum. Thus $w_i'(1) = 0$, that is,

$$(p' - p)\cdot D_p x(p',w)(p' - p) = 0.$$

This is a contradiction to the negative definiteness.

Case 2: $w_i(\lambda) > 0$ for some $\lambda \in [0,\bar{\lambda}]$.

Then, by the mean-value theorem, there exists $\lambda^* \in (0,\lambda)$ such that $w_i(\lambda) - w_i(0) = w_i'(\lambda^*)(\lambda - 0)$. By $w_i(0) = 0$ and $w_i(\lambda) > 0$, $w_i'(\lambda^*) > 0$. That is,

$$(p' - p)\cdot D_p x(p(\lambda^*),w)(p' - p) > 0.$$

This is a contradiction to the negative definiteness. Our proof is thus completed.

4.C.2 If $D_p x_i(p,\alpha_i w)$ is negative definite on the whole $\mathbb{R}^L$ for every i, then the sum $\sum_i D_p x_i(p,\alpha_i w)$ is negative definite on the whole $\mathbb{R}^L$. Since $D_p x(p,w) = \sum_i D_p x_i(p,\alpha_i w)$, $D_p x(p,w)$ is negative definite on the whole $\mathbb{R}^L$, implying that $x(p,w)$ satisfies the ULD property. To establish the WA, one way is simply to notice that the ULD property implies the WA, as the latter considers only compensated price changes.

Another way is to prove the given differential sufficient condition. Let's assume that $w > 0$. Define $H = \{v \in \mathbb{R}^L: v\cdot x(p,w) = 0\}$, that is, H is the hyperplane with normal $x(p,w)$ that goes through the origin. Then $p \notin H$ because $p\cdot x(p,w) = w > 0$. Thus, if $v \in \mathbb{R}^L$ and v is not proportional to p, then there exist $v_1 \in \mathbb{R}^L$ and $v_2 \in \mathbb{R}^L$ such that $v_1 \in H$, $v_1 \neq 0$, v_2 is proportional to p, and $v = v_1 + v_2$. Since $S(p,w)v_2 = 0$ and $v_2 \cdot S(p,w) = 0$ by Proposition 2.F.3, we have

$$\begin{aligned} v \cdot S(p,w)v &= (v_1 + v_2) \cdot S(p,w)(v_1 + v_2) \\ &= v_1 \cdot S(p,w)v_1 + v_1 \cdot S(p,w)v_2 + v_2 \cdot S(p,w)v_1 + v_2 \cdot S(p,w)v_2 \\ &= v_1 \cdot S(p,w)v_1. \end{aligned}$$

But here, by $v_1 \in H$,

$$S(p,w)v_1 = (D_p x(p,w) + D_w x(p,w) x(p,w)^T)v_1 = D_p x(p,w)v_1.$$

Hence $v_1 \cdot S(p,w)v_1 = v_1 \cdot D_p x(p,w)v_1 < 0$ because $v_1 \neq 0$ and $D_p x(p,w)$ is negative definite. Thus the WA holds.

4.C.3 A Giffen good will be a most familiar example. In the figure below, good 1 is a Giffen good.

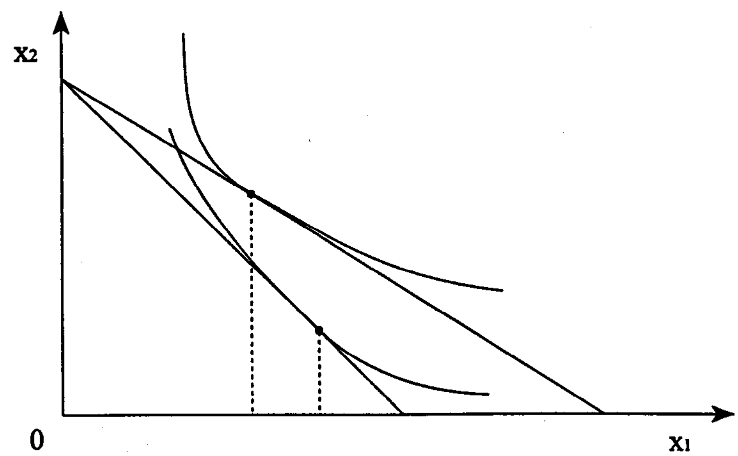


Figure 4.C.3

This example shows that the ULD property is actually not derived from the utility maximization. It is a restriction on preferences.

4.C.4 The L-shaped indifference curves imply that for every strictly positive price vector, the consumer's demand is always be at the corner of a upper contour set. Hence no compensated price change will change the demand. Thus $S_i(p,w_i) = 0$ for every (p,w_i) and $D_p x_i(p,w_i) = D_{w_i} x_i(p,w_i) x_i(p,w_i)^T$. Suppose that there exists (p,w_i) such that $D_{w_i} x_i(p,w_i) \neq (1/w_i) x_i(p,w_i)$. Since $p \cdot D_{w_i} x_i(p,w_i) = p \cdot (1/w_i) x(p,w_i) = 1$, this implies that $D_{w_i} x_i(p,w_i)$ and $x_i(p,w_i)$ are not proportional. Hence there exists a $v \in \mathbb{R}^2$ such that $v \cdot D_{w_i} x_i(p,w_i) < 0$ and $v \cdot x_i(p,w_i) > 0$, as illustrated in the figure below:

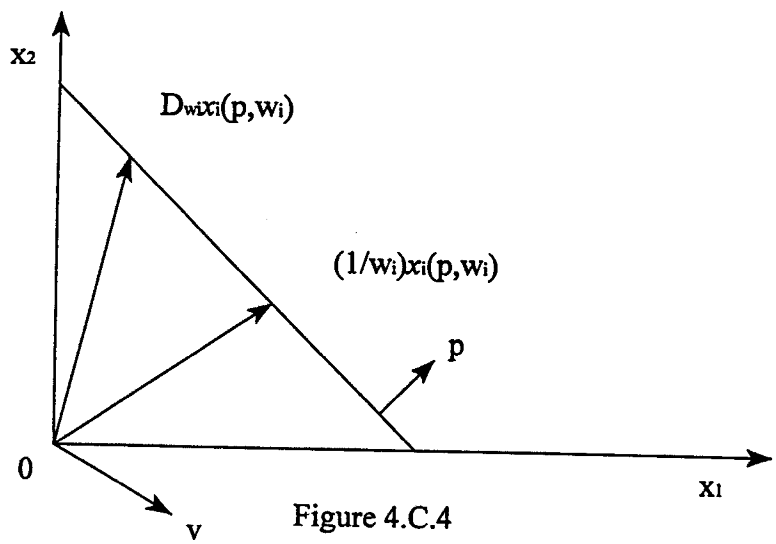


Figure 4.C.4

Thus

$$v \cdot D_p x_i(p,w_i) v = - v \cdot D_{w_i} x_i(p,w_i) x_i(p,w_i)^T v = - (v \cdot D_{w_i} x_i(p,w_i))(v \cdot x_i(p,w_i)) > 0,$$

which implies that the ULD property is not satisfied. Thus, if it is in fact satisfied, then we must have $D_{w_i} x_i(p,w_i) = (1/w_i) x_i(p,w_i)$ for every (p,w_i). Thus the unique wealth expansion path, which is the set of the corners of the upper contour sets, is a ray going through the origin. Hence the preference is homothetic.

4.C.5 Following the hint, we fix w = 1 and write $x_i(p) = x_i(p,1)$. Consider the indirect demand function $g_i(x) = \frac{1}{x\cdot\nabla u_i(x)}\nabla u_i(x)$. Since $x_i(p) = x$ if and only if $g_i(x) = p$, the ULD property of $x_i(p)$ is equivalent to the following property: if $x \neq y$, then $(g_i(x) - g_i(y))\cdot(x - y) < 0$. For this latter property, it is sufficient to show that $D^2 g_i(x)$ is negative definite. We shall now establish this.

By the chain rule (Appendix M.A),

$$Dg_i(x) = (x\cdot\nabla u_i(x))^{-2}((x\cdot\nabla u_i(x))D^2u_i(x) - \nabla u_i(x)\nabla u_i(x)^T - \nabla u_i(x)x^T D^2u_i(x))$$

Let $q = \nabla u_i(x)$ and $C = D^2u_i(x)$, then this can be rewritten as

$$Dg_i(x) = (x\cdot q)^{-2}((x\cdot q)C - qq^T - qx^T C).$$

We need to show that $v\cdot Dg_i(x)v < 0$ for every $v \neq 0$. If $v\cdot q = 0$, then $v\cdot Dg_i(x)v = (x\cdot q)^{-1}v\cdot Cv < 0$. (This property is equivalent to the negative definiteness of the bordered Hessian of $u_i(\cdot)$ and used to guarantee the differentiability of the demand function, as explained in the Appendix to Chapter 3). So suppose that $v\cdot q \neq 0$. By multiplying a scalar to v if necessary, we can assume that $v\cdot q = x\cdot q$. Then

$$v\cdot Dg_i(x)v = (x\cdot q)^{-1}(v\cdot Cv - v\cdot q - x\cdot Cv).$$

By $x\cdot q > 0$, we need to show that $v\cdot Cv - v\cdot q - x\cdot Cv < 0$. Since C is symmetric,

$$v\cdot Cv - x\cdot Cv = (v - (1/2)x)\cdot C(v - (1/2)x) - (1/4)x\cdot Cx.$$

Since $u_i(\cdot)$ is concave, C is negative semidefinite and the first term in the above expression is non-positive. Thus,

$$v\cdot Cv - x\cdot Cv \le -(1/4)x\cdot Cx.$$

Hence

$$v\cdot Cv - v\cdot q - x\cdot Cv \le -(1/4)x\cdot Cx - q\cdot x.$$

Since $-\frac{x\cdot Cx}{q\cdot x} < 4$, the right-hand side is negative. Hence so is the left-hand side.

4.C.6 By differentiating both sides of $u(\lambda x) = \lambda u(x)$ with respect to λ and taking $\lambda = 1$, we obtain $\nabla u(x)\cdot x = u(x)$. Then by differentiating both sides of this equality with respect to x, we obtain $D^2u(x)x + \nabla u(x) = \nabla u(x)$. Thus $D^2u(x)x = 0$ and hence $\sigma(x) = 0$.

4.C.7 Suppose that the distribution of wealth has a differentiable, nonincreasing density function $f(\cdot)$ over the interval $[0,\bar{w}]$. Let $v \in \mathbb{R}^L$ and $v \neq 0$, then, just as in the proof of Proposition 4.C.4, we have

$$v\cdot Dx(p)v = \int_0^{\bar{w}}(v\cdot S(p,w)v)f(w)dw - \int_0^{\bar{w}}(v\cdot D_w\tilde{x}(p,w))(v\cdot\tilde{x}(p,w))f(w)dw.$$

Here, the first term is negative, unless v is proportional to p. (This property is equivalent to the negative definiteness of the bordered Hessian of $u_i(\cdot)$ and used to guarantee the differentiability of the demand function, as explained in the Appendix to Chapter 3). As for the second term, just as in the proof of Proposition 4.C.4,

$$\int_0^{\bar{w}}(v\cdot D_w\tilde{x}(p,w))(v\cdot\tilde{x}(p,w))f(w)dw = (1/2)\int_0^{\bar{w}}\frac{d(v\cdot\tilde{x}(p,w))^2}{dw}f(w)dw.$$

By integration by parts and $\tilde{x}(p,0) = 0$, this is equal to

$$(1/2)(v\cdot\tilde{x}(p,\bar{w}))^2 f(\bar{w}) - (1/2)\int_0^{\bar{w}}(v\cdot x(p,w))^2 f'(w)dw$$

The first part of this is always nonnegative, and it is positive when v is proportional to p. The integral of the second part is nonpositive because $f'(w) \le 0$. Thus $(1/2)\int_0^{\bar{w}}\frac{d(v\cdot\tilde{x}(p,w))^2}{dw}f(w)dw \ge 0$. Hence $v\cdot Dx(p)v < 0$.

To see that there are unimodal density functions for which the conclusions of this propositions do not hold, recall that

$$v\cdot Dx(p)v = \int_0^{\bar{w}}(v\cdot S(p,w)v)f(w)dw - \int_0^{\bar{w}}(v\cdot D_w\tilde{x}(p,w))(v\cdot\tilde{x}(p,w))f(w)dw.$$

To be specific, let $v = (1,0,\dots,0) \in \mathbb{R}^L$, then

$$D_1 x_1(p) = \int_0^{\bar{w}} s_{11}(p,w)f(w)dw - \int_0^{\bar{w}} D_w\tilde{x}_1(p,w)\tilde{x}_1(p,w)f(w)dw.$$

Suppose also that the graph of the function $w \to \tilde{x}_1(p,w)$ is as shown below.

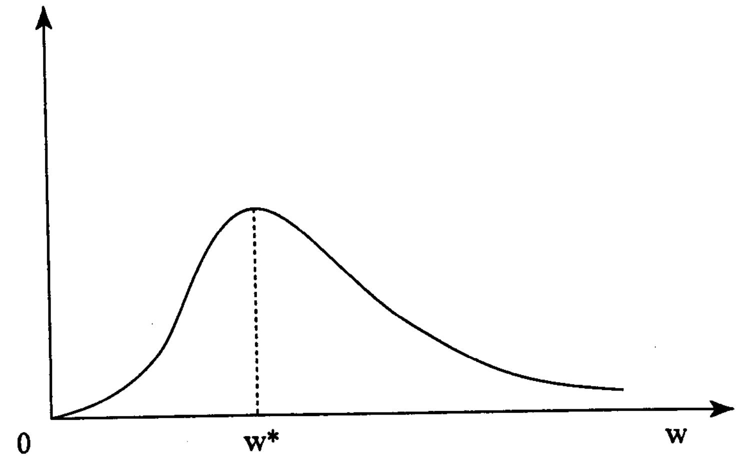


Figure 4.C.7

Then $D_w\tilde{x}_1(p,w) \geq 0$ for every $w \in [0,w^*]$ and $D_w\tilde{x}_1(p,w) \leq 0$ for every $w \in [w^*,\bar{w}]$. Thus $\int_0^{w^*} D_w\tilde{x}_1(p,w)\tilde{x}_1(p,w)f(w)dw \geq 0$ and $\int_{w^*}^{\bar{w}} D_w\tilde{x}_1(p,w)\tilde{x}_1(p,w)f(w)dw \leq 0$. If the density function $f(\cdot)$ is nonincreasing, the weight placed on the interval $[0,w^*]$ is sufficient to dominate the negative effect of the interval $[w^*,\bar{w}]$, implying that $\int_0^{\bar{w}} D_w\tilde{x}_1(p,w)\tilde{x}_1(p,w)f(w)dw \geq 0$. If the distribution function is not nonincreasing, then the weight on the interval $[w^*,\bar{w}]$ may dominate that on the interval $[0,w^*]$, so that $\int_0^{\bar{w}} D_w\tilde{x}_1(p,w)\tilde{x}_1(p,w)f(w)dw < 0$. It could even dominate the substitution effect, in which case we have

$$v \cdot Dx(p)v = \int_0^{\bar{w}} (v \cdot S(p,w)v)f(w)dw - \int_0^{\bar{w}} (v \cdot D_w\tilde{x}(p,w))(v \cdot \tilde{x}(p,w))f(w)dw > 0.$$

4.C.8 By substituting (4.C.6) into (4.C.5), we obtain

$$S(p,w) = \sum_i S_i(p,\alpha_i w) - \sum_i D_{w_i} x_i(p,\alpha_i w)x_i(p,\alpha_i w) + (\sum_i \alpha_i D_{w_i} x_i(p,\alpha_i w))x(p,w)^T.$$

On the other hand, the right-hand side of (4.C.7) equals to

$$\sum_i S_i(p,\alpha_i w) - \sum_i D_{w_i} x_i(p,\alpha_i w)x_i(p,\alpha_i w) + (\sum_i \alpha_i D_{w_i} x_i(p,\alpha_i w))x(p,w)^T$$

$$+ D_w x(p,w)(\sum_i x_i(p,\alpha_i w))^T - (\sum_i \alpha_i) D_w x(p,w)) x(p,w)^T$$

$$= \sum_i S_i(p,\alpha_i w) - \sum_i D_{w_i} x_i(p,\alpha_i w) x_i(p,\alpha_i w) + (\sum_i \alpha_i D_{w_i} x_i(p,\alpha_i w)) x(p,w)^T.$$

We have thus proved (4.C.8).

4.C.9 The homotheticity implies that $D_{w_i} x_i(p,\alpha_i w) = (1/\alpha_i w) x_i(p,\alpha_i w)$ and hence $D_w x(p,w) = \sum_i \alpha_i D_{w_i} x_i(p,\alpha_i w) = (1/w) x(p,w)$. Thus

$$C(p,w) = \sum_i \alpha_i ((1/\alpha_i w) x_i(p,\alpha_i w) - (1/w) x(p,w))((1/\alpha_i) x_i(p,\alpha_i w) - x(p,w))^T$$

$$= \sum_i (\alpha_i / w)((1/\alpha_i) x_i(p,\alpha_i w) - x(p,w))((1/\alpha_i) x_i(p,\alpha_i w) - x(p,w))^T.$$

Thus, for every $v \in \mathbb{R}^L$,

$$v \cdot C(p,w) v = \sum_i (\alpha_i / w)(((1/\alpha_i) x_i(p,\alpha_i w) - x(p,w)) \cdot v)^2 \geq 0$$

Therefore, $C(p,w)$ is positive semidefinite.

4.C.10 Let's start by formulating our continuum-of-consumers situation. We take the interval [0,2] as the set of (the names of) the consumers. The population density is equal to 1/2 uniformly on [0,2]. We assume that the proportion of each consumer's wealth to the average wealth (which is, in this situation, a counterpart of the aggregate wealth) is constant regardless of the amount of the aggregate wealth. We further assume that when the "average" wealth is equal to $\bar{w} > 0$, the wealth of consumer $\eta \in [0,2]$ is equal to $\eta\bar{w}$. Since $\int_0^2 \eta\bar{w}(1/2)d\eta = \bar{w}$, the term "average" is justified.

The average demand is then defined by $x(p,\bar{w}) = \int_0^2 \tilde{x}(p,\eta\bar{w})(1/2)d\eta$ and the Slutsky matrix $S(p,\bar{w})$ is defined as in (4.C.4). The Slutsky matrix of consumer η is denoted by $\tilde{S}(p,\eta\bar{w})$. We then define $C(p,\bar{w})$ by

$$C(p,\bar{w}) = \int_0^2 \tilde{S}(p,\eta\bar{w})(1/2)d\eta - S(p,\bar{w}).$$

Just as in the small-type discussion following the proof of Proposition 4.C.4, we have

$$C(p,\bar{w}) = \int_0^2(\eta/2)(D_w\tilde{x}(p,\eta\bar{w}) - D_w x(p,\bar{w}))((1/\eta)\tilde{x}(p,\eta\bar{w}) - x(p,\bar{w}))^T d\eta$$
$$= \int_0^2(1/2)D_w\tilde{x}(p,\eta\bar{w})\tilde{x}(p,\eta\bar{w})d\eta - D_w x(p,\bar{w})x(p,\bar{w})^T$$
$$= \int_0^2(1/2)D_w\tilde{x}(p,\eta\bar{w})\tilde{x}(p,\eta\bar{w})d\eta - (\int_0^2(\eta/2)D_w\tilde{x}(p,\eta\bar{w})d\eta)x(p,\bar{w})^T.$$

Thus, for each $v \in \mathbb{R}^L$,

$$v\cdot C(p,\bar{w})v$$
$$= \int_0^2(1/2)(v\cdot D_w\tilde{x}(p,\eta\bar{w}))(v\cdot\tilde{x}(p,\eta\bar{w}))d\eta - (\int_0^2(\eta/2)v\cdot D_w\tilde{x}(p,\eta\bar{w})d\eta)(v\cdot x(p,\bar{w})).$$

For the first term, we know that

$$\int_0^2(1/2)(v\cdot D_w\tilde{x}(p,\eta\bar{w}))(v\cdot\tilde{x}(p,\eta\bar{w}))d\eta = (1/4\bar{w})(v\cdot\tilde{x}(p,2\bar{w}))^2.$$

As for the second term, by integration by parts,

$$\int_0^2(\eta/2)v\cdot D_w\tilde{x}(p,\eta\bar{w})d\eta$$
$$= [(\eta/2)(v\cdot\tilde{x}(p,\eta\bar{w}))(1/\bar{w})]_{\eta=0}^{\eta=2} - \int_0^2(1/2)(v\cdot\tilde{x}(p,\eta\bar{w}))(1/\bar{w})d\eta$$
$$= (1/\bar{w})(v\cdot\tilde{x}(p,2\bar{w})) - (1/\bar{w})(v\cdot x(p,\bar{w})).$$

Hence

$$v\cdot C(p,\bar{w})v = (1/\bar{w})((1/4)(v\cdot\tilde{x}(p,2\bar{w}))^2 - (v\cdot\tilde{x}(p,2\bar{w}))(v\cdot x(p,\bar{w})) + (v\cdot x(p,\bar{w}))^2)$$
$$= (1/\bar{w})((1/2)(v\cdot\tilde{x}(p,2\bar{w})) - v\cdot x(p,\bar{w}))^2 \geq 0.$$

Thus $C(p,\bar{w})$ is positive semidefinite.

4.C.11 **(a)** When deriving individual demands from the first-order conditions of utility maximization, we will neglect the nonnegativity constraints (which is investigated in Exercise 3.D.4(c)). In fact, we will later see that, for prices and wealths under consideration, the demands are always in the interior of the nonnegative orthant.

It follows directly from the first-order conditions that

$$x_1(p,w/2) = (x_{11}(p,w/2),x_{21}(p,w/2)) = (w/2p_1 - 4p_1/p_2,\ 4p_1^2/p_2^2),$$
$$x_2(p,w/2) = (x_{12}(p,w/2),x_{22}(p,w/2)) = (4p_2^2/p_1^2,\ w/2p_2 - 4p_2/p_1),$$

Hence

$$x(p,w) = x_1(p,w/2) + x_2(p,w/2)$$
$$= (w/2p_1 - 4p_1/p_2 + 4p_2^2/p_1^2,\ w/2p_2 - 4p_2/p_1 + 4p_1^2/p_2^2).$$

(b) Denote the (ℓ,k) entry of the Slutsky matrix $S_i(p,w)$ of consumer i by $s_{\ell ki}(p,w)$. Since $\partial x_{21}(p,w/2)/\partial w_1 = 0$, $s_{221}(p,w/2) = \partial x_{21}(p,w/2)/\partial p_2 = - 8p_1^2/p_2^3$. Hence by Proposition 2.F.3, $s_{211}(p,w/2) = s_{121}(p,w/2) = 8p_1/p_2^2$, and hence $s_{111}(p,w/2) = - 8/p_2$. Thus

$$S_1(p,w/2) = \begin{bmatrix} - 8/p_2 & 8p_1/p_2^2 \\ 8p_1/p_2^2 & - 8p_1^2/p_2^3 \end{bmatrix}.$$

Similarly, we can show that

$$S_2(p,w/2) = \begin{bmatrix} - 8p_1^2/p_2^3 & 8p_2/p_1^2 \\ 8p_2/p_1^2 & - 8/p_1 \end{bmatrix}.$$

We can also apply Proposition 2.F.3 to derive the Slutsky matrix $S(p,w)$ of the aggregate demand function:

$$S(p,w) = \begin{bmatrix} - w/4p_1^2 - 6/p_2 - 6p_2^2/p_1^3 & w/4p_1p_2 + 6p_1/p_2^2 + 6p_2/p_1^2 \\ w/4p_1p_2 + 6p_1/p_2^2 + 6p_2/p_1^2 & - w/4p_2^2 - 6/p_1 - 6p_1^2/p_2^3 \end{bmatrix}.$$

By Exercise 2.F.9(b) (and $S(p,w)p = 0$), if $dp \in \mathbb{R}^2$, $dp \neq 0$, and dp is not proportional to p, then $dp \cdot S(P,w)dp < 0$. Thus, according to the small-type discussion after Proposition 2.F.3, the aggregate demand function $x(p,w)$ satisfies the WA.

(c) By substituting $p = (1,1)$, we obtain

$$C(p,w) = \textstyle\sum_i S_i(p,w/2) - S(p,w) = \begin{bmatrix} w/4 - 4 & 4 - w/4 \\ 4 - w/4 & w/4 - 4 \end{bmatrix}.$$

Thus, if $w > 16$, then it is positive semidefinite, and if $8 < w < 16$, then it is negative semidefinite. (This can be shown by applying Theorem M.D.2(ii), or by noticing that $C(p,w)p = 0$ and $p \cdot C(p,w) = 0$ and then applying the argument in the proof of Exercise 2.F.9(b).) For example, if w = 12 and v = (1,0), then $v \cdot C(p,w)v = -1$. Thus $C(p,w)$ is not positive semidefinite. We saw in (b) that the aggregate demand function satisfies the WA. We can thus conclude that in order for an aggregate demand function to satisfy the WA, it is not necessary that the matrix $C(p,w)$ is positive semidefinite.

(d) Here is a figure depicting the wealth expansions paths of the two consumers for p = (1,1). They intersect each other at (4,4) because $x_1(1,1,8) = x_2(1,1,8) = (4,4)$. Note that if $8 < w < 16$, the Engel curves resemble those of figure 4.C.2(b); if $w > 16$, the Engel curves resemble those of figure 4.C.2(a).

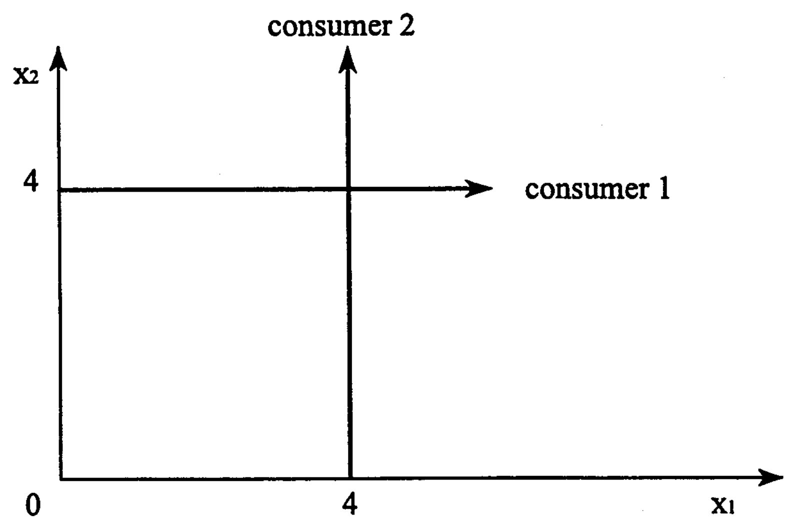


Figure 4.C.11(d)

4.C.12 As suggested in the hint, our example is a two-commodity, two-consumer economy in which the two consumers have the same preference and the wealth distribution rule is such that when the aggregate wealth is equal to 4, the wealth of consumer 1 is equal to 1 and the wealth of consumer 2 is equal to 3. The example here is essentially the same as Example 4.C.1 (which is illustrated in Figure 4.C.1). Example 4.C.1 is not directly applicable to the present context, because the two consumers have the same wealth but the different demands. However, if wealth levels are different between the two consumers, then it is possible that they have the same preference and yet demands similar to those in Figure 4.C.1, yielding a violation of the WA in the aggregate. This is illustrated in the figure below:

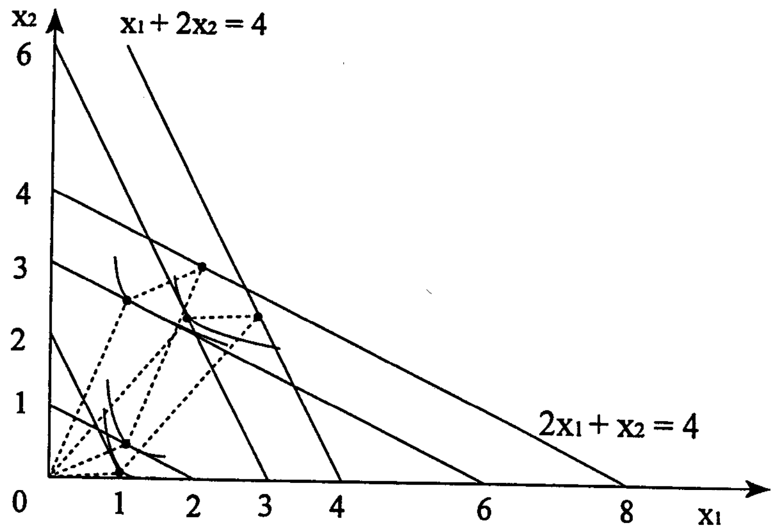


Figure 4.C.12

4.C.13 We consider a two-commodity, two-consumer economy with the given wealth distribution rule $w_1(p,w) = wp_1/(p_1 + p_2)$ and $w_2(p,w) = wp_2/(p_1 + p_2)$. The preferences of the consumers are represented by the following utility functions:

$$u_1(x_1) = \text{Min}\ \{x_{11}, 2x_{21}\},$$
$$u_2(x_2) = \text{Min}\ \{2x_{12}, x_{22}\}.$$

So the preferences are homothetic and have L-shaped indifference curves. The unique wealth expansion paths are depicted in the figure below:

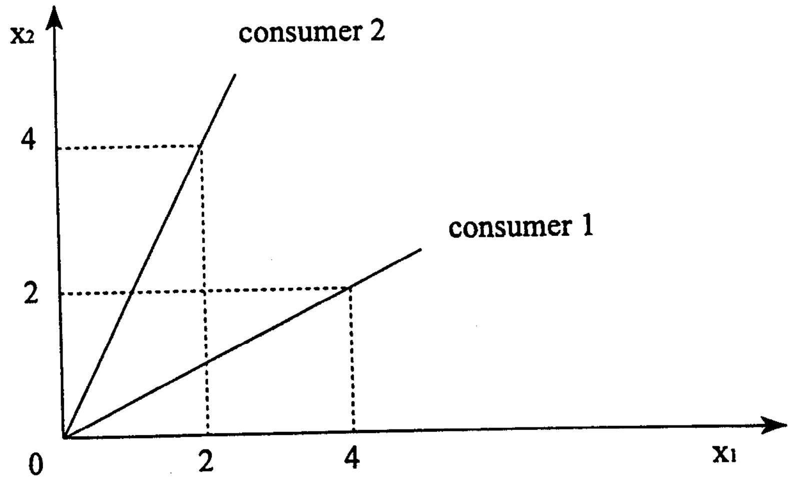


Figure 4.C.13

Just as we saw in the answer to Exercise 3.D.5(c), the individual demand functions are

$$x_1(p,w_1) = (2w_1/(2p_1 + p_2),\ w_1/(2p_1 + p_2)),$$

$$x_2(p,w_2) = (w_2/(p_1 + 2p_2),\ 2w_2/(p_1 + 2p_2)).$$

The aggregate demand function is given by

$$x(p,w) = x_1(p,w_1(p,w)) + x_2(p,w_2(p,w)).$$

We claim that the aggregate demand function does not satisfy the WA. To see this, define $\Delta = \{p \in \mathbb{R}^2_{++}: p_1 + p_2 = 1\}$ and our restrict attention to $p \in \Delta$ and $w = 1$. Then $w_i(p,w) = p_i$ for both i and hence

$$\lim_{p\to(1,0)} x(p,1) = \lim_{p\to(1,0)} x_1(p,w_1(p,1)) = (1,1/2),$$

$$\lim_{p\to(0,1)} x(p,1) = \lim_{p\to(0,1)} x_2(p,w_2(p,1)) = (1/2,1).$$

Thus, if $p \in \Delta$, $q \in \Delta$, and p_2 and q_1 are sufficiently small, then $q\cdot x(p,1) < 1$ and $p\cdot x(q,1) < 1$. Hence the aggregate demand function does not satisfy the WA.

Proposition 4.C.1 does not apply to this example because the wealth distribution rule of this example depends on prices, while the proposition does not allow this.

4.D.1 It is easy to check from the budget constraints that the distribution $(x_1(p,w_1(p,w)),\ldots,x_I(p,w_I(p,w)))$ of commodity bundles satisfies the constraints of the maximization problem in this exercise, and, from the definition of the indirect utility functions, that it attains the value $v(p,w)$. It thus remains to show that if $(x_1,\ldots,x_I)$ satisfies the constraints of the maximization problem in this exercise, then $W(u_1(x_1),\ldots,u_I(x_I)) \le v(p,w)$. For each i, define $w_i = p\cdot x_i$, then $\sum_i w_i = \sum_i p\cdot x_i = p\cdot(\sum_i x_i) \le w$, that is, $(w_1,\ldots,w_I)$ satisfies the constraint of the maximization problem of (4.D.1). Hence, by the definition of $v(p,w)$,

$$W(v_1(p,w_1),\ldots,v_I(p,w_I)) \le v(p,w).$$

On the other hand, by the definition of the indirect utility functions, $u_i(x_i) \le v_i(p,w_i)$ for every i. This and the increasingness of $W(\cdot)$ imply that

$$W(u_1(x_1),\ldots,u_I(x_I)) \le W(v_1(p,w_1),\ldots,v_I(p,w_I)).$$

Hence $W(u_1(x_1),\ldots,u_I(x_I)) \le v(p,w)$. This completes our proof.

4.D.2 To check that $v(p,w)$ is increasing in w, let p be a price vector, w and w' be two aggregate wealth levels with $w \le w'$, and $(w_1,\ldots,w_I)$ be a solution to the social welfare maximization problem of (p,w). Then

$$v(p,w) = W(v_1(p,w_1),\ldots,v_I(w_I)).$$

Also, $\sum_i w_i \le w$ and hence $\sum_i w_i \le w'$. Thus, by the definition of $v(p,w')$,

$$W(v_1(p,w_1),\ldots,v_I(p,w_I)) \le v(p,w').$$

Hence $v(p,w) \le v(p,w')$.

To check that $v(p,w)$ is nonincreasing in p, let w be an aggregate wealth level, p and p' be two price vectors with and $p' \ge p$, and let $(w_1,\ldots,w_I)$ be a solution to the social welfare maximization problem of (p',w). Then

$$v(p',w) = W(v_1(p',w_1),\ldots,v_I(p',w_I)).$$

Also, $v_i(p',w_i) \le v_i(p,w_i)$ for every i because $p' \ge p$. Since $W(\cdot)$ is increasing, this implies that

$$W(v_1(p',w_1),\ldots, v_I(p',w_I)) \le W(v_1(p,w_1),\ldots,v_I(p,w_I)).$$

By the definition of $v(p,w)$,

$$W(v_1(p,w_1),\ldots,v_I(p,w_I)) \le v(p,w).$$

Hence $v(p',w) \le v(p,w)$.

To verify the homogeneity of degree zero and the quasiconvexity, we apply the equivalence of the two maximization problems established in Exercise 4.D.1. For any (p,w) and $\lambda > 0$, the two price-wealth pairs (p,w) and $(\lambda p,\lambda w)$ give the same constraints to the maximization problem of Exercise 4.D.1. Hence $v(p,w) = v(\lambda p,\lambda w)$, implying the homogeneity. As for the quasiconvexity, let $v \in \mathbb{R}$. Let (p,w) and (p',w') be two price-wealth pairs such that $v(p,w) \le v$ and $v(p',w') \le v$. Let $\lambda \in [0,1]$ and define $p'' = \lambda p + (1 - \lambda)p'$ and $w'' = \lambda w + (1 - \lambda)w'$. Let $(x_1,\ldots,x_I)$ be a solution for (p",w"). Then $p''\cdot(\sum_i x_i) \le w''$ and hence

$$\lambda p\cdot(\sum_i x_i) + (1 - \lambda)p'\cdot(\sum_i x_i) \le \alpha w + (1 - \alpha)w'.$$

We must thus have either $w \ge p\cdot(\sum_i x_i)$ or $w' \ge p'\cdot(\sum_i x_i)$. Hence we must have either $W(u_1(x_1),\ldots,u_I(x_I)) \le v(p,w)$ or $W(u_1(x_1),\ldots,u_I(x_I)) \le v(p',w')$. In either case, we have $W(u_1(x_1),\ldots,u_I(x_I)) \le v$. Hence $v(p,w)$ is quasiconvex.

4.D.3 The welfare maximization problem is now rewritten with nonnegativity constraints:

$$\max_{(w_1,\ldots,w_I)} W(v_1(p,w_1),\ldots,v_I(p,w_I))$$
$$\text{s.t.} \quad \sum_i w_i \le w,$$
$$w_i \ge 0 \text{ for all i}.$$

We assume that $x_i(p,0) = 0$ for every i and every $p \gg 0$. This assumption is satisfied if, for example, the consumption sets are all $\mathbb{R}^L_+$. Then $\nabla_p v_i(p,0) = 0$ for every i and every p >> 0. The Kuhn-Tucker conditions for the social welfare maximization (Theorem M.K.2) are that there exist $\lambda > 0$ and $\mu_i \geq 0$ such that

$$(\partial W/\partial u_i)(\partial v_i/\partial w_i) - \lambda + \mu_i = 0 \text{ for every } i = 1,\ldots,I,$$

$$\textstyle\sum_i w_i = w,$$

$$w_i \geq 0 \text{ and } \mu_i w_i = 0 \quad \text{for every } i = 1,\ldots,I,$$

where all derivatives are evaluated at the solution

$$(w_1,\ldots,w_I) = (w_1(p,w),\ldots,w_I(p,w)).$$

The Envelop Theorem (Theorem M.L.1) implies that

$$\partial v/\partial w = \lambda,$$

$$\partial v/\partial p_\ell = \textstyle\sum_i(\partial W/\partial u_i)(\partial v_i/\partial p_\ell).$$

Now define $J = \{i\colon w_i(p,w) > 0\}$. Since $\mu_i = 0$ for every $i \in J$, $\partial v/\partial w = (\partial W/\partial u_i)(\partial v_i/\partial w_i)$ for every $i \in J$. Since $\partial v_i/\partial p_\ell = 0$ for any $i \notin J$, $\partial v/\partial p_\ell = \sum_{i\in J}(\partial W/\partial u_i)(\partial v_i/\partial p_\ell)$. Thus,

$$-\frac{\partial v/\partial p_\ell}{\partial v/\partial w} = -\sum_{i\in J}\frac{(\partial W/\partial u_i)(\partial v_i/\partial p_\ell)}{\partial v/\partial w} = -\sum_{i\in J}\frac{(\partial W/\partial u_i)(\partial v_i/\partial p_\ell)}{(\partial W/\partial u_i)(\partial v_i/\partial w_i)}$$

$$= -\sum_{i\in J}\frac{\partial v_i/\partial p_\ell}{\partial v_i/\partial w_i} = \textstyle\sum_{i\in J}x_i(p,w_i(p,w)) = \sum_i x_i(p,w_i(p,w)).$$

Hence the Walrasian demand function derived from $v(p,w)$ equals $\sum_i x_i(p,w_i(p,w))$ and the proof is completed.

4.D.4 **(a)** If the $u_i(\cdot)$ and $W(\cdot)$ are monotone, then so is $u(\cdot)$.

If the $u_i(\cdot)$ and $W(\cdot)$ are continuous, then the composite map $(x_1,\ldots,x_I) \to W(u_1(x_1),\ldots,u_I(x_I))$ is also continuous. Thus, by the Theorem of the Maximum (Theorem M.K.6), $u(\cdot)$ is also continuous.

If the $u_i(\cdot)$ are concave and $W(\cdot)$ are monotone and concave, then $u(\cdot)$ is concave. This can be proved as follows. Let $x \in \mathbb{R}^L$, $x' \in \mathbb{R}^L$, and $\lambda \in [0,1]$. Define $x'' = \lambda x + (1-\lambda)x' \in \mathbb{R}^L$. Let $(x_1,\dots,x_I) \in \mathbb{R}^{LI}$, $(x'_1,\dots,x'_I) \in \mathbb{R}^{LI}$, $\sum_i x_i = x$, $\sum_i x'_i = x'$, $u(x) = W(u_1(x_1),\dots,u_I(x_I))$, and $u(x') = W(u_1(x'_1),\dots,u_I(x'_I))$. Define $(x''_1,\dots,x''_I) \in \mathbb{R}^{LI}$ by $x''_i = \lambda x_i + (1-\lambda)x'_i$ for each i. Then $\sum_i x''_i \le \lambda x + (1-\lambda)x'$ and hence

$$u(\lambda x + (1-\lambda)x') \ge W(u_1(x''_1),\dots,u_I(x''_I)).$$

By the concavity of the $u_i(\cdot)$ and the monotonicity of $W(\cdot)$,

$$W(u_1(x''_1),\dots,u_I(x''_I))$$
$$\ge W(\lambda u_1(x_1) + (1-\lambda)u_1(x'_1),\ \dots,\ \lambda u_I(x_I) + (1-\lambda)u_I(x'_I)).$$

By the concavity of $W(\cdot)$,

$$W(\lambda u_1(x_1) + (1-\lambda)u_1(x'_1),\ \dots,\ \lambda u_I(x_I) + (1-\lambda)u_I(x'_I))$$
$$\ge \lambda u(x) + (1-\lambda)u(x').$$

Hence $u(\lambda x + (1-\lambda)x') \ge \lambda u(x) + (1-\lambda)u(x')$. Hence $u(\cdot)$ is concave.

It is worthwhile to point out that the quasiconcavity of $W(\cdot)$ and the $u_i(\cdot)$ does not imply that of $u(\cdot)$. As an example, let L = 2, I = 2, $u_1(x_1) = x_{11}^2$, $u_2(x_2) = x_{22}^2$, and $W(u_1,u_2) = u_1 + u_2$. Then

$$W(2,0) = W(u_1(2,0),u_2(0,0)) = W(4,0) = 4,$$
$$W(0,2) = W(u_1(0,0),u_2(0,2)) = W(0,4) = 4,$$
$$W(1,1) = W(u_1(1,0),u_2(0,1)) = W(1,1) = 2.$$

(b) We shall first prove that, for every $x \in \mathbb{R}^L$, if there exists $(x_1,\dots,x_I) \in \mathbb{R}^{LI}$ that satisfies $\sum_i x_i \le x$ and is a solution to the maximization problem of Exercise 4.D.1, then x is a solution to the maximization problem of this part. In fact, suppose that $(x_1,\dots,x_I) \in \mathbb{R}^{LI}$, $\sum_i x_i \le x$, and $(x_1,\dots,x_I)$ is a solution to the maximization problem of Exercise 4.D.1. Let $x' \in \mathbb{R}^L$ and $p \cdot x' \le w$. Let $(x'_1,\dots,x'_I) \in \mathbb{R}^{LI}$, $\sum_i x'_i \le x'$, and $u(x') = W(u_1(x'_1),\dots,u_I(x'_I))$. Then $p \cdot (\sum_i x'_i) \le w$. Hence $W(u_1(x'_1),\dots,u_I(x'_I)) \le W(u_1(x_1),\dots,u_I(x_I))$. Since

$\sum_i x_i \le x$, $W(u_1(x_1),\ldots,u_I(x_I)) \le u(x)$. Thus $u(x') \le u(x)$. Hence x is a solution to the maximization problem of this part.

By Exercise 4.D.1, $(x_1(p,w_1(p,w)),\ldots,x_I(p,w_I(p,w))$ is a solution to its maximization problem. Thus, by the above result, $\sum_i x_i(p,w_i(p,w))$ is a solution to the maximization problem of this part. Hence the Walrasian demand function generated from it is equal to the aggregate demand function.

4.D.5 There is no positive representative consumer if the WA is violated. Example 4.C.1 thus serves as an example for this exercise.

4.D.6 The social welfare maximization problem is now written as

$$\text{Max}_{(w_1,\ldots,w_I)} \ \sum_i \alpha_i \ln v_i(p,w_i)$$
$$\text{s.t.} \qquad \sum_i w_i \le w.$$

The first-order conditions are that there exists $\lambda > 0$ such that $(\partial W/\partial u_i)(\partial v_i/\partial w_i) = \lambda$ for every i, where all derivatives are evaluated at a solution $(w_1,\ldots,w_I)$. By the definition of $W(\cdot)$, $\partial W/\partial u_i = \alpha_i/v_i(p,w_i)$. By Exercise 3.D.3(b), $v_i(p,w_i)$ is homogeneous of degree one in w_i and hence $\partial v_i(p,w_i)/\partial w_i = v_i(p,w_i)/w_i$. Hence the left-hand sides of the above first-order conditions equal $(\alpha_i/v_i(p,w_i))(v_i(p,w_i)/w_i) = \alpha_i/w_i$ for every i. Thus $w_i = \alpha_i/\lambda$. Since $\sum_i \alpha_i = 1$ and $\sum_i w_i = w$, $w = 1/\lambda$. Hence $w_i = \alpha_i w$. Thus $w_i(p,w) = \alpha_i w$.

4.D.7 The social welfare maximization problem is now written as

$$\text{Max}_{(w_1,\ldots,w_I)} \ \sum_i a_i(p) + b(p)(\sum_i w_i)$$
$$\text{s.t.} \qquad \sum_i w_i \le w.$$

Thus any $(w_1,\ldots,w_I)$ with $\sum_i w_i = w$ is a solution to this problem and $v(p,w) =$

$\sum_i a_i(p) + b(p)w$.

4.D.8 Suppose that (p',w') passes the potential compensation test over (p,w). Then there exists $(w'_1,\ldots,w'_I) \in \mathbb{R}^I$ such that $\sum_i w'_i \le w$ and $v_i(p',w'_i) > v_i(p,w_i(p,w))$. Since $W(\cdot)$ is increasing, $W(v_1(p',w'_1),\ldots,v_I(p',w'_I)) > v(p,w)$. By the definition of $v(p',w')$, $v(p',w') \ge W(v_1(p',w'_1),\ldots,v_I(p',w'_I))$. Hence $v(p',w') > v(p,w)$.

4.D.9 Define $A_i = \{x_i: u_i(x_i) \ge u_i(\bar{x}_i)\}$, then $A = \sum_i A_i$. Then, for every $p \gg 0$, $x \in \mathbb{R}^L$ is a solution to

$$\text{Min}_{y \in A}\ p \cdot y$$

if and only if there exists $(x_1,\ldots,x_I) \in A_1 \times \ldots \times A_I$ such that $\sum_i x_i = x$ and, for every i, x_i is a solution to

$$\text{Min}_{y_i \in A_i}\ p \cdot x_i.$$

Hence $\text{Min}\{p \cdot y: y \in A\} = \sum_i \text{Min}\{p \cdot y_i: y_i \in A_i\}$. By the definition,

$$\text{Min}\{p \cdot y_i: y_i \in A_i\} = e_i(p,u_i(\bar{x}_i))$$

for every i. Hence

$$\text{Min}\{p \cdot y: y \in A\} = \sum_i e_i(p,u_i(\bar{x}_i)).$$

On the other hand, by the definition,

$$\text{Min}\{p \cdot y: y \in B\} = e(p,u(\bar{x})).$$

Since $A \subset B$, $\text{Min}\{p \cdot y: y \in A\} \ge \text{Min}\{p \cdot y: y \in B\}$. Hence

$$g(p) = e(p,u(\bar{x})) - \sum_i e_i(p,u_i(\bar{x}_i)) \le 0$$

for every p. By the definition, $g(\bar{p}) = 0$. Hence, by the second-order necessary condition for a maximum (Section M.K), $D^2 g(\bar{p})$ is negative semidefinite. Since $D^2 g(\bar{p}) = S(p,u(\bar{x})) - \sum_i S_i(p,u_i(\bar{x}_i))$ by Propositions 3.G.3 and 3.G.3, $S(p,u(\bar{x})) - \sum_i S_i(p,u_i(\bar{x}_i))$ is negative semidefinite.

4.D.10 As we saw in the answer to Exercise 4.C.11(b), for every (p,w), $S(p,w)$ is symmetric, and if $dp \in \mathbb{R}^2$, $dp \neq 0$, and dp is not proportional to p, then $dp \cdot S(P,w)dp < 0$. Thus, according to Section 3.H, there exists a positive representative consumer. But according to Exercise 4.C.11(c), if $8 < w < 16$, then the matrix $C(p,w)$ is not positive semidefinite. Hence, according to the small-type discussion in Section 4.D, there is no normative representative consumer.

4.D.11 We shall give an example in which L = 3 and $C_{12}(p,\bar{w}) \neq C_{21}(p,\bar{w})$. By the definition,

$$\begin{aligned} &C_{12}(p,\bar{w}) \\ &\quad = \int_0^2 (1/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)\tilde{x}_2(p,\eta\bar{w})d\eta - (\int_0^2 (\eta/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)d\eta)\tilde{x}_2(p,\eta\bar{w}). \end{aligned}$$

Here, by integration by parts,

$$\begin{aligned} &\int_0^2 (\eta/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)d\eta \\ &= \int_0^2 (\eta/2\bar{w})(d\tilde{x}_1(p,\eta\bar{w})/d\eta)d\eta \\ &= [(\eta/2\bar{w})\tilde{x}(p,\eta\bar{w})]_{\eta=0}^{\eta=2} - \int_0^2 (1/2\bar{w})\tilde{x}_1(p,\eta\bar{w})d\eta \\ &= (1/\bar{w})\tilde{x}(p,2\bar{w}) - (1/2\bar{w})x_1(p,\bar{w}). \end{aligned}$$

Hence

$$\begin{aligned} C_{12}(p,\bar{w}) = &\int_0^2 (1/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)\tilde{x}_2(p,\eta\bar{w})d\eta \\ &- (1/\bar{w})\tilde{x}_1(p,2\bar{w})x_2(p,\bar{w}) + (1/2\bar{w})x_1(p,\bar{w})x_2(p,\bar{w}). \end{aligned}$$

Similarly,

$$\begin{aligned} C_{21}(p,\bar{w}) = &\int_0^2 (1/2)\tilde{x}_1(p,\eta\bar{w})(\partial\tilde{x}_2(p,\eta\bar{w})/\partial w)d\eta \\ &- (1/\bar{w})x_1(p,\bar{w})\tilde{x}_2(p,2\bar{w}) + (1/2\bar{w})x_1(p,\bar{w})x_2(p,\bar{w}). \end{aligned}$$

Again by integration by parts,

$$\int_0^2 (1/2)\tilde{x}_1(p,\eta\bar{w})(\partial\tilde{x}_2(p,\eta\bar{w})/\partial w)d\eta$$

$$= \int_0^2 (1/2)\tilde{x}_1(p,\eta\bar{w})(1/\bar{w})(d\tilde{x}_2(p,\eta\bar{w})/d\eta)d\eta$$

$$= [(1/2\bar{w})\tilde{x}_1(p,\eta\bar{w})\tilde{x}_2(p,\eta\bar{w})]_{\eta=0}^{\eta=2} - \int_0^2 (1/2\bar{w})(d\tilde{x}_1(p,\eta\bar{w})/d\eta)\tilde{x}_2(p,\eta\bar{w})d\eta$$

$$= (1/2\bar{w})\tilde{x}_1(p,2\bar{w})\tilde{x}_2(p,2\bar{w}) - \int_0^2 (1/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)\tilde{x}_2(p,\eta\bar{w})d\eta$$

It is thus sufficient to obtain an example in which

$$\int_0^2 (1/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)\tilde{x}_2(p,\eta\bar{w})d\eta - (1/\bar{w})\tilde{x}_1(p,2\bar{w})x_2(p,\bar{w})$$

$$\neq (1/2\bar{w})\tilde{x}_1(p,2\bar{w})\tilde{x}_2(p,2\bar{w}) - \int_0^2 (1/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)\tilde{x}_2(p,\eta\bar{w})d\eta$$

$$- (1/\bar{w})x_1(p,\bar{w})\tilde{x}_2(p,2\bar{w}),$$

or, equivalently,

$$\int_0^2 (1/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)\tilde{x}_2(p,\eta\bar{w})d\eta$$

$$\neq (1/2\bar{w})\tilde{x}_1(p,2\bar{w})\tilde{x}_2(p,2\bar{w}) + (1/\bar{w})\tilde{x}_1(p,2\bar{w})x_2(p,\bar{w}) - (1/\bar{w})x_1(p,\bar{w})\tilde{x}_2(p,2\bar{w}).$$

So consider a preference, a price vector p, and the average wealth $\bar{w}$ such that:

$$\tilde{x}_\ell(p,2\bar{w}) = 2 \text{ for both } \ell = 1,2,$$

$$x_\ell(p,\bar{w}) = \int_0^2 (1/2)\tilde{x}_\ell(p,\eta\bar{w})d\eta = 1 \text{ for both } \ell = 1,2.$$

(Another restriction will be given shortly. The demand for good 3 is determined by Walras' law.) Then the right-hand side of the above inequality is equal to $2/\bar{w}$. It is then sufficient to show that

$$\int_0^2 (1/2)(\partial\tilde{x}_1(p,\eta\bar{w})/\partial w)\tilde{x}_2(p,\eta\bar{w})d\eta \neq 2/\bar{w}.$$

But if the graphs of the functions $\eta \to \tilde{x}_\ell(p,\eta\bar{w})$ are as in the figure below, then the first term can be made as close to zero as needed. The above inequality can thus be established.

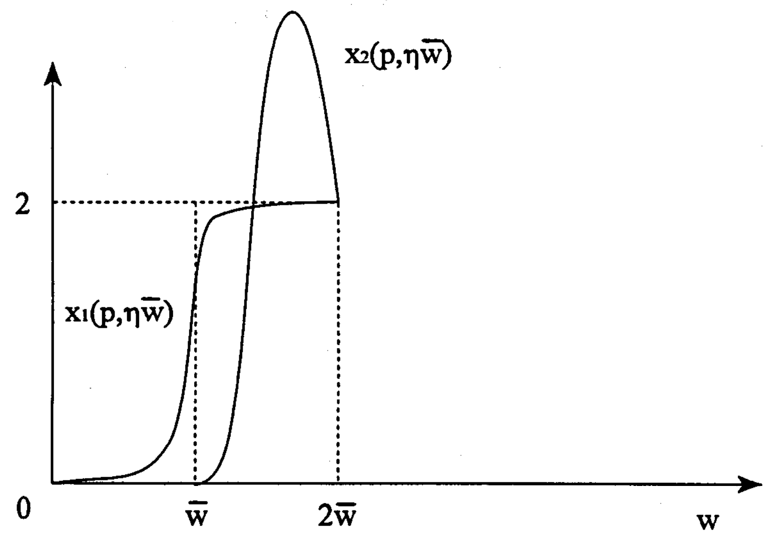


Figure 4.D.11

CHAPTER 5

5.B.1 The first example violates irreversibility and the second one satisfies this property.

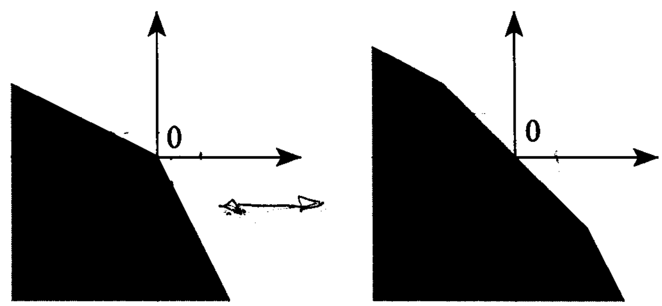


Figure 5.B.1

5.B.2 Suppose first that Y exhibits constant returns to scale. Let $z \in \mathbb{R}_+^{L-1}$ and $\alpha > 0$. Then $(-z, f(z)) \in Y$. By the constant returns to scale, $(-\alpha z, \alpha f(z)) \in Y$. Hence $\alpha f(z) \leq f(\alpha z)$. By applying this inequality to αz in place of z and α^{-1} in place of α, we obtain $\alpha^{-1} f(\alpha z) \leq f(\alpha^{-1}(\alpha z)) = f(z)$, or $f(\alpha z) \leq \alpha f(z)$. Hence $f(\alpha z) = \alpha f(z)$. The homogeneity of degree one is thus obtained.

Suppose conversely that $f(\cdot)$ is homogeneous of degree one. Let $(-z, q) \in Y$ and $\alpha \geq 0$, then $q \leq f(z)$ and hence $\alpha q \leq \alpha f(z) = f(\alpha z)$. Since $(-\alpha z, f(\alpha z)) \in Y$, we obtain $(-\alpha z, \alpha q) \in Y$. The constant returns to scale is thus obtained.

5.B.3 Suppose first that Y is convex. Let $z, z' \in \mathbb{R}_+^{L-1}$ and $\alpha \in [0,1]$, then

$(-z, f(z)) \in Y$ and $(-z', f(z')) \in Y$. By the convexity,

$$(-(\alpha z + (1-\alpha)z), \alpha f(z) + (1-\alpha)f(z)) \in Y.$$

Thus, $\alpha f(z) + (1-\alpha)f(z) \leq f(\alpha z + (1-\alpha)z)$. Hence $f(z)$ is concave.

Suppose conversely that $f(z)$ is concave. Let $(q, -z) \in Y$, $(q', -z') \in Y$, and $\alpha \in [0,1]$, then $q \leq f(z)$ and $q' \leq f(z')$. Hence

$$\alpha q + (1-\alpha)q' \leq \alpha f(z) + (1-\alpha)f(z').$$

By the concavity,

$$\alpha f(z) + (1-\alpha)f(z') \leq f(\alpha z + (1-\alpha)z').$$

Thus

$$\alpha q + (1-\alpha)q' \leq f(\alpha z + (1-\alpha)z').$$

Hence

$$(-(\alpha z + (1-\alpha)z'), \alpha q + (1-\alpha)q') = \alpha(-z, q) + (1-\alpha)(-z', q') \in Y.$$

Therefore Y is convex.

5.B.4 Note first that if Y itself is additive, then $Y^+ = Y$ by the definition of the additive closure. This applies to Figures 5.B.4, 5.B.6(a), 5.B.6(b), 5.B.7, and 5.B.8. So we shall not depict them.

Let's now take up the cases in which the production set is not additive. Note first that Y^+ is equal to the set of vectors of $\mathbb{R}^L$ that can be represented as the sum of finitely many vectors of Y. If the production set is convex, as in Figures 5.B.1, 5.B.2(a), 5.B.2(b), 5.B.3(a), 5.B.3(b), and 5.B.5(a) (which is the same as 5.B.2(b)), then we have the following stronger property: Y^+ consists of all "multiplied" production plans. To be more precise, for each positive integer n, define $nY \subset \mathbb{R}^L$ by $nY = \{ny \in \mathbb{R}^L: y \in Y\}$. We then claim that $Y^+ = \cup_{n=1}^{\infty} nY$. In fact, if $y \in nY$ for some n, then $(1/n)y \in Y$. By the definition, $(1/n)y \in Y^+$ and hence $y = n((1/n)y) \in Y^+$. Thus $Y^+ \supset \cup_{n=1}^{\infty} nY$. To prove that $Y^+ \subset \cup_{n=1}^{\infty} nY$, it is sufficient to show that $\cup_{n=1}^{\infty} nY$ is additive, by the definition of Y^+. So let $y \in \cup_{n=1}^{\infty} nY$ and $y' \in \cup_{n=1}^{\infty} nY$. Then there exist positive integers n and n' such that $y \in nY$ and $y' \in n'Y$. Thus $(1/n)y \in Y$ and $(1/n')y' \in Y$. Since Y is convex and $n/(n + n') + n'/(n + n') = 1$,

$$(n/(n + n'))((1/n)y) + (n'/(n + n'))((1/n')y') \in Y.$$

That is, $(1/(n + n'))(y + y') \in Y$. Thus $y + y' \in (n + n')Y$.

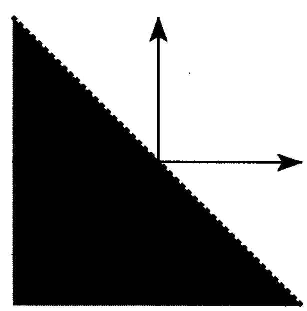
If the production set is not convex, as in Figure 5.B.5(b), then we no longer have $Y^+ = \cup_{n=1}^{\infty} nY$. As we can see in the above proof, while we still have $Y^+ \supset \cup_{n=1}^{\infty} nY$, we need not have $Y^+ \subset \cup_{n=1}^{\infty} nY$. That is, it may be true that some production plans in Y^+ can be attained only by allocating different inputs to different production units. This point can be formulated as follows. Let (y_1^*, y_2^*) be the production plan at which the average return is maximized. That is, $y_2^*/|y_1^*| > y_2/|y_1|$ for any other $y = (y_1, y_2) \in Y$. Assume that the function that associates each $y_1 < 0$ to the average return at y_1 is quasiconcave. (This appears to be true from the figure. It is equivalent to saying that the average return function is single-peaked.) Let $y_1 < 0$ be an (aggregate) input level and n be the positive integer such that $ny_1^* < y_1 \le (n - 1)y_1^*$. Let $(y_{11}, \ldots, y_{1J})$ be an output-maximizing allocation of aggregate input y_1. (The number J of production units to be used is also being optimized here.) We shall prove that one of the following three cases must then apply:

(i) $J = n - 1$ and $y_{1j} = y_1/(n - 1)$ for every j.

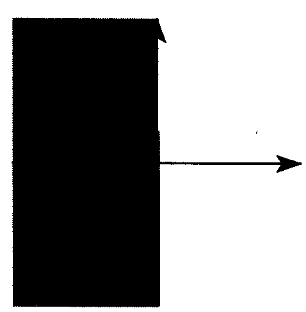
(ii) $J = n$ and $y_{1j} = y_1/n$ for every j.

(iii) $J = n$ and there exists $k \in \{1, \ldots, J\}$ such that $y_{1k} = y_1 - (n - 1)y_1^*$ and $y_{1j} = y_1^*$ for any $j \ne k$.

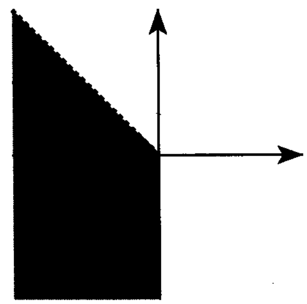
To prove this, note first that we must have either $y_{1j} \ge y_1^*$ for every j, or $y_{1j} \le y_1^*$ for every j. In fact, if neither of these applies, then a small input reassignment from a production unit with $y_{1j} < y_1^*$ to one with $y_{1j} > y_1^*$ increases the (aggregate) output level, because the average return increases. If $y_{1j} \le y_1^*$ for every j, then the average return is decreasing at every y_{1j} and hence the y_{1j} must be all the same. By the quasiconcavity, J must be the maximum integer that satisfies $y_1^* = \sum_j y_{1j}$ (and $y_{1j} \le y_1^*$). Hence J = n - 1 and (i) applies. If $y_{1j} \ge y_1^*$ for every j, then the average return is increasing at every y_{1j}. If the y_{1j} are all the same, then, by the quasiconvexity, J must be the minimum integer that satisfies $y_1^* = \sum_j y_{1j}$ (and $y_{1j} \ge y_1^*$). Hence J = n and (ii) applies. So suppose that some of the y_{1j} are different. If there exist two production units for which $y_{1j} > y_1^*$, then a small input reassignment from one to the other would increase the output level, because the average return is increasing. Hence there must exist at most one k for which $y_{1k} > y_1^*$. By the quasiconcavity, $y_{1j} = y_1^*$ for any $j \ne k$. Thus (iii) applies. This last case happens when the average return decreases very fast after the input level goes beyond y_1^*.



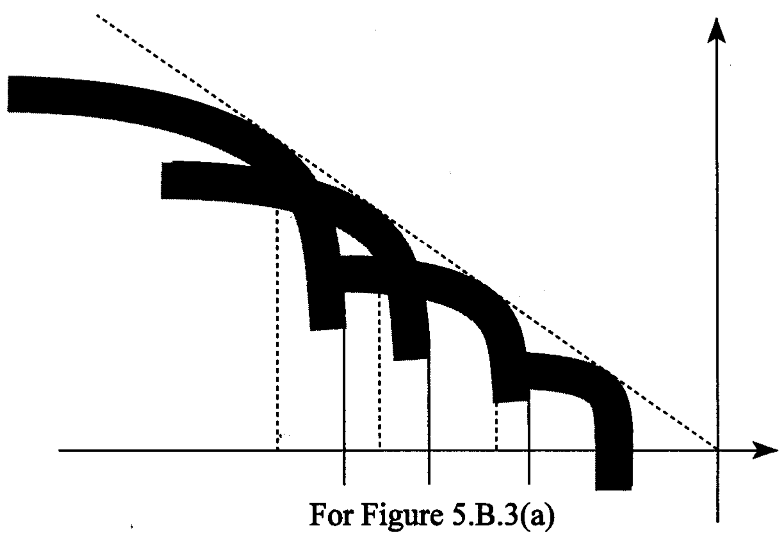
For Figure 5.B.1



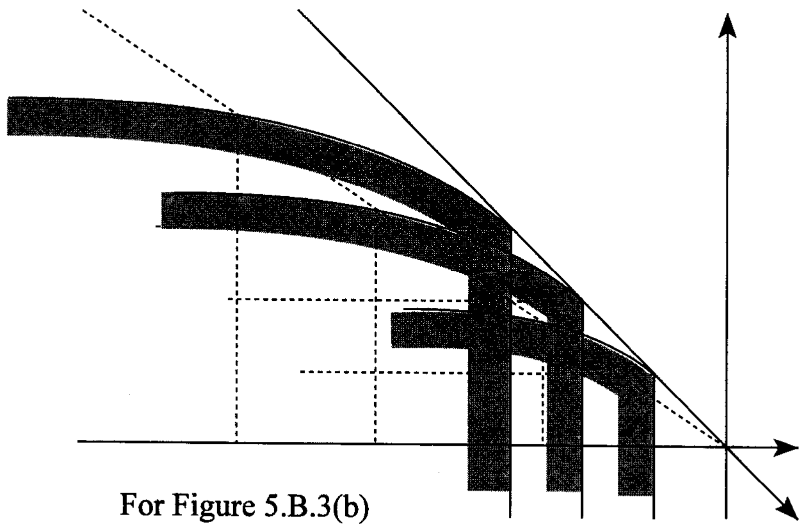
For Figure 5.B.2(a)



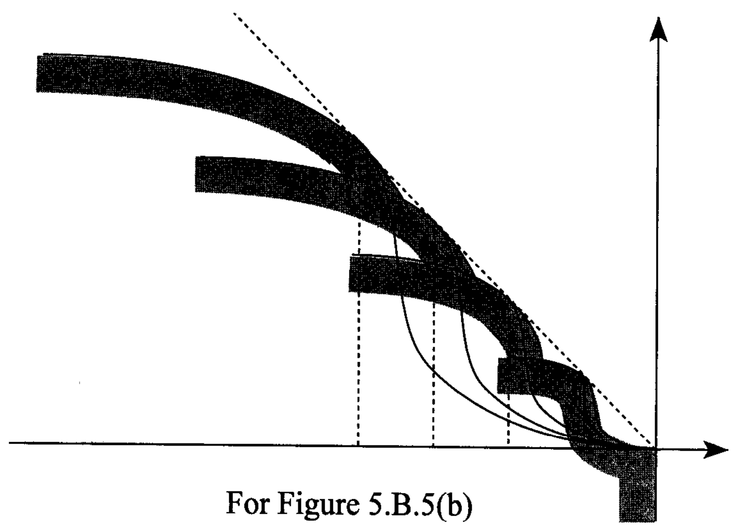
For Figure 5.B.2(b)



For Figure 5.B.3(a)



For Figure 5.B.3(b)



For Figure 5.B.5(b)

5.B.5 Let $y \in Y$ and $v \in -\mathbb{R}^L_+$. Then, for every $n \in \mathbb{N}$, $nv \in -\mathbb{R}^L_+$ and hence $nv \in Y$ by $Y \subset -\mathbb{R}^L_+$. Since Y is convex,

$$(1 - 1/n)y + (1/n)(nv) = (1 - 1/n)y + v \in Y.$$

Since Y is closed, $y + v = \lim_{n\to\infty} ((1 - 1/n)y + v) \in Y$.

5.B.6 **(a)** From the given functions $\phi_i(\cdot)$ $(i = 1,2)$, the production set is defined as $Y = \{(y_1,y_2,q)$: there exist $q_1 \geq 0$ and $q_2 \geq 0$ such that $q_1 + q_2 \geq q$ and $-y_i \geq \phi_i(q_i)$ for both $i\}$. A three-dimensional production set is depicted in the following picture, assuming that the $\phi_i(\cdot)$ are convex.

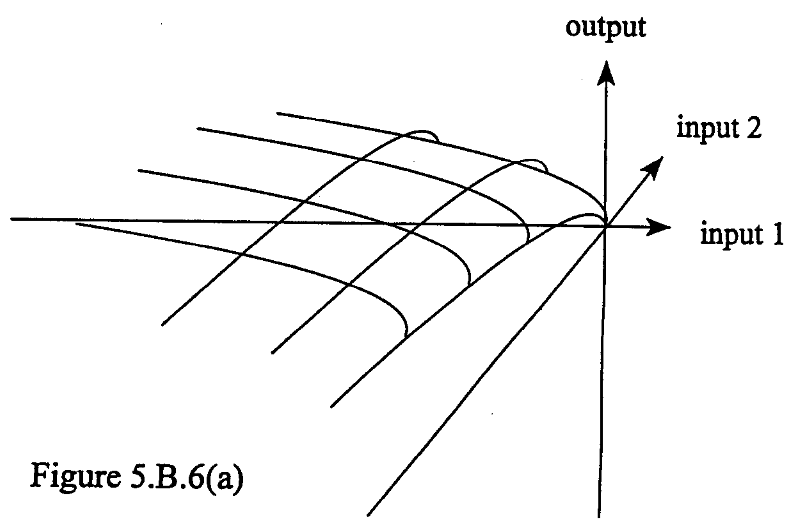


Figure 5.B.6(a)

(b) We claim that the condition that

$$\phi_i(q_i + q_i') \leq \phi_i(q_i) + \phi_i(q_i') \text{ for all } q_i \geq 0,\ q_i' \geq 0, \text{ and } i = 1,2,$$

is sufficient for additivity. In fact, let $(y_1,y_2,q) \in Y$ and $(y_1',y_2',q') \in Y$. Then there exist (q_1,q_2) such that $q_1 + q_2 \geq q$ and $-y_i \geq \phi_i(q_i)$, and (q_1', q_2') such that $q_1' + q_2' \geq q'$ and $-y_i' \geq \phi_i(q_i')$. Then

$$(q_1 + q_1') + (q_2 + q_2') \geq q + q',$$

and

$$-(y_i + y_i') \geq \phi_i(q_i) + \phi_i(q_i') \geq \phi_i(q_i + q_i').$$

Thus $(y_1 + y'_1, y_2 + y'_2, q + q') \in Y$, establishing the additivity.

(c) Let the output price be p. The first-order necessary conditions for profit maximization are that, for both i, $w_i\phi'_i(q_i) \le p$, with equality if $q_i > 0$. The interpretation is that marginal cost (in monetary term) due to a unit increase of output level must be smaller than or equal to the output price, and the former must be equal to the latter if the output level is positive.

If both $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are convex (so that the corresponding production set is convex), then, according to Theorem M.K.3, these first-order necessary conditions are also sufficient.

(d) Let $q > 0$. By renumbering $i = 1,2$ if necessary, we can assume that $w_1\phi_1(q) \le w_2\phi_2(q)$. In order to prove the statement, it is sufficient to show that, for any $q_1 > 0$ and $q_2 > 0$ with $q_1 + q_2 = q$, we have $w_1\phi_1(q_1) + w_2\phi_2(q_2) > w_1\phi_1(q)$. In fact, since the $\phi_i(\cdot)$ are strictly concave (and $\phi_i(0) = 0$) and $q_1/q + q_2/q = 1$,

$$w_1\phi_1(q_1) + w_2\phi_2(q_2) > w_1(q_1/q)\phi_1(q) + w_2(q_2/q)\phi_2(q)$$
$$= (q_1/q)w_1\phi_1(q) + (q_2/q)w_2\phi_2(q) \ge w_1\phi_1(q).$$

The statement is thus proved. The strict concavity of the $\phi_i(\cdot)$ is interpreted as the increasing returns to scale, which makes the statement quite plausible: Under the increasing returns to scale, it is better to concentrate on one technique.

The isoquants of the input use is drawn in the following figure. Note that the additive separability imposes the same restriction on the isoquants as that alluded to in Exercise 3.G.4(b).

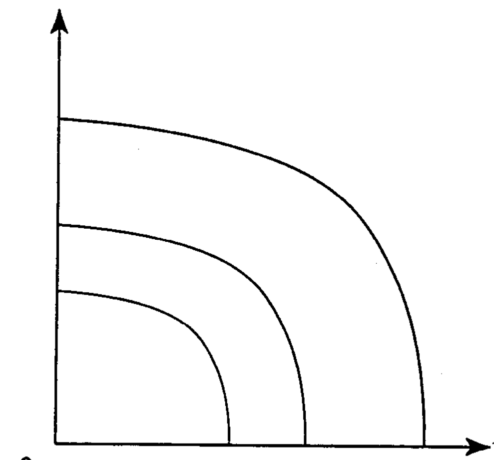


Figure 5.B.6(d)

5.C.1 If there is a production plan $y \in Y$ with $p \cdot y > 0$, then, by using $\alpha y \in Y$ with a large $\alpha > 1$, it is possible to attain any sufficiently large profit level. Hence $\pi(p) = \infty$. If, on the contrary, $p \cdot y \leq 0$ for all $y \in Y$, then $\pi(p) \leq 0$. Thus we have either $\pi(p) = +\infty$ or $\pi(p) \leq 0$.

5.C.2 Let $p \gg 0$, $p' \gg 0$, $\alpha \in [0,1]$, and $y \in y(\alpha p + (1 - \alpha)p')$, then $p \cdot y \leq \pi(p)$ and $p' \cdot y \leq \pi(p')$. Thus,

$$(\alpha p + (1 - \alpha)p') \cdot y = \alpha p \cdot y + (1 - \alpha)p' \cdot y \leq \alpha\pi(p) + (1 - \alpha)\pi(p').$$

Since $(\alpha p + (1 - \alpha)p') \cdot y = \pi(\alpha p + (1 - \alpha)p')$,

$$\pi(\alpha p + (1 - \alpha)p') \leq \alpha\pi(p) + (1 - \alpha)\pi(p').$$

Hence $\pi(\cdot)$ is convex.

5.C.3 The homogeneity of $c(\cdot)$ in q is implied by that of $z(\cdot)$. We shall thus prove this latter homogeneity only. Let $w \gg 0$, $q \geq 0$, and $\alpha > 0$. Let $z \in z(w,q)$. Since $f(\cdot)$ is homogeneous of degree one, $f(\alpha z) = \alpha f(z) \geq \alpha q$. For every $z' \in \mathbb{R}^{L-1}_{+}$, if $f(z') \geq \alpha q$, then $f(\alpha^{-1}z') = \alpha^{-1}f(z') \geq q$. Thus, by $z \in$

$z(w,q)$, $w \cdot (\alpha^{-1}z') \geq w \cdot z$. Hence $w \cdot z' \geq w \cdot (\alpha z)$. Thus $\alpha z \in z(w,\alpha q)$. So $\alpha z(w,q) \subset z(w,\alpha q)$. By applying this inclusion to α^{-1} in place of α and αq in place of q, we obtain $\alpha^{-1}z(w,\alpha q) \subset z(w,\alpha^{-1}(\alpha q))$, or $z(w,\alpha q) \subset \alpha z(w,q)$ and thus conclude that $z(w,\alpha q) = \alpha z(w,q)$.

We next prove property (viii). Let $w \in \mathbb{R}^{L-1}_{++}$, $q \geq 0$, $q' \geq 0$, and $\alpha \in [0,1]$. Let $z \in z(w,q)$ and $z' \in z(w,q')$. Then $f(z) \geq q$, $f(z') \geq q'$, $c(w,q) = w \cdot z$, and $c(w,q') = w \cdot z'$. Hence

$$\alpha c(w,q) + (1 - \alpha)c(w,q') = \alpha(w \cdot z) + (1 - \alpha)(w \cdot z') = w \cdot (\alpha z + (1 - \alpha)z').$$

Since $f(\cdot)$ is concave,

$$f(\alpha z + (1 - \alpha)z') \geq \alpha f(z) + (1 - \alpha)f(z') \geq \alpha q + (1 - \alpha)q'.$$

Thus $w \cdot (\alpha z + (1 - \alpha)z') \geq c(w, \alpha q + (1 - \alpha)q')$. That is,

$$\alpha c(w, q) + (1 - \alpha)c(w, q') \geq c(w, \alpha q + (1 - \alpha)q').$$

5.C.4 [First printing errata: When there are multiple outputs, the function $f(z)$ need not be well defined because it is conceivably possible to produce different combinations of outputs from a single combination of inputs. Assuming that the first L - M commodities are inputs and the last M commodities are outputs, we should thus understand the set $\{z \geq 0: f(z) \geq q\}$ as $\{z \in \mathbb{R}^{L-M}_{+}: (- z, q) \in Y\}$.] For each $q \geq 0$, define

$$Y(q) = \{z \in \mathbb{R}^{L-M}_{+}: (- z, q) \in Y\} = \{z \in \mathbb{R}^{L-M}_{+}: f(z) \geq q\}.$$

Then $c(\cdot,q)$ is the support function of Y(q) for every q. Hence property (ii) follows from the discussion of Section 3.F. Moreover, according to Exercise 3.F.1, if Y(q) is closed and convex, then

$$Y(q) = \{z \in \mathbb{R}^{L-M}_{+}: w \cdot z \geq c(w,q) \text{ for all } w \gg 0\}.$$

Since $Y = \{(- z, q): q \geq 0 \text{ and } z \in Y(q)\}$, this implies property (iii).

To prove property (iv), let $w \gg 0$, $q \geq 0$, $\alpha > 0$, $z \in z(w,q)$, and $z' \in Y(q)$. Then $w \cdot z \leq w \cdot z'$. Hence $(\alpha w) \cdot z \leq (\alpha w) \cdot z'$. Thus $z \in z(\alpha w,q)$. Therefore

$z(w,q) \subset z(\alpha w,q)$. By applying this inclusion to αw in place of w and α^{-1} in place of α, we obtain $z(\alpha w,q) \subset z(\alpha^{-1}(\alpha w),q) = z(w,q)$. Property (iv) thus follows.

Property (iv) implies the homogeneity of degree one of $c(\cdot)$ in w, which is the first part of property (i). As for its second part, let $w >> 0$, $q \geq 0$, $q' \geq 0$, and $q' \geq q$. Then $Y(q') \subset Y(q)$. Since $c(\cdot,q)$ and $c(\cdot,q')$ are the support functions, this inclusion implies that $c(w,q') \geq c(w,q)$. Hence $c(\cdot)$ is nondecreasing in q.

As for property (v), note that $z(w,q) = Y(q) \cap \{z \in \mathbb{R}^L: w \cdot z = c(w,q)\}$ for every $w >> 0$ and $q \geq 0$. Since both of the two sets on the right-hand side is convex, so is the intersection, and hence so is $z(w,q)$. As for the single-valuedness, let $q \geq 0$, $w >> 0$, $z \in z(w,q)$, $z' \in z(w,q)$, and $z \neq z'$. Also suppose that $Y(q)$ is strictly convex. By the convexity of $z(w,q)$,

$$(1/2)z + (1/2)z' \in z(w,q).$$

By the strict convexity of $Y(q)$, there exists a $z'' \in Y(q)$ such that

$$(1/2)z + (1/2)z' >> z''.$$

Hence $w\cdot((1/2)z + (1/2)z') > w\cdot z''$, which contradicts $(1/2)z + (1/2)z' \in z(w,q)$. Thus $z(w,q)$ must be single-valued.

Property (vi) follows from the fact that $c(\cdot,q)$ is the support function of $Y(q)$ and the duality theorem (Proposition 3.F.1).

Property (vi) implies that if $z(\cdot,q)$ is differentiable at $\bar{w}$, then $D_w z(\bar{w},q) = D_w(\nabla_w c(\bar{w},q)) = D_w^2 c(\bar{w},q)$. As a Hessian matrix, this is symmetric. By property (ii), it is negative semidefinite. By property (iv), $D_w z(\bar{w},q)\bar{w} = 0$. Property (vii) is thus established.

5.C.5 If the production function $f(\cdot)$ is quasiconcave, then the set $Y(q) = \{z \in \mathbb{R}_+^{L-1}: f(z) \geq q\}$ is convex for any q, and thus property (iii) holds. (Note that we used the convexity of Y(q), not of Y.)

If there is a single input and the production function is given by $f(z) = z^2$, then it is quasiconcave but the corresponding production set exhibits increasing returns to scale. Quasiconcavity is thus compatible with increasing returns.

5.C.6 Throughout the following answers, the input prices are denoted by $w \gg 0$ and the output price is denoted by $p > 0$. For convenience, we denote by $z(p,w)$ the input demands at prices (p,w). As a preliminary result, by using the implicit function theorem (Theorem M.E.1), we shall prove that $z(\cdot)$ is a continuously differentiable function and give its derivatives in terms of $f(\cdot)$. (To be rigorous, we need to assume that $f(\cdot)$ is twice continuously differentiable and the input demands are always strictly positive, so that the nonnegativity constraints never bind.)

Since $D^2 f(z)$ is negative definite for all z, the first-order necessary and sufficient condition (Theorems M.K.2 and M.K.3) for profit maximization is then that z is an input demand vector at prices (p,w) if and only if

$$p\nabla f(z) - w = 0.$$

If we regard the left-hand side as defining the function defined over (p,w,z), then the function is continuously differentiable and its derivative with respect to z is equal to $pD^2 f(z)$. It is negative definite, and hence has the inverse matrix (at every z). Thus, by the implicit function theorem (Theorem M.E.1), for each (p,w), there is a unique z for which $p\nabla f(z) - w = 0$ and the mapping from (p,w) to z is continuously differentiable. This is equivalent to saying that $z(\cdot)$ is a continuously differentiable function. The implicit function theorem also tells us that

$$\partial z(p,w)/\partial p = (-\ 1/p)D^2 f(z(p,w))^{-1}\nabla f(z(p,w)),$$

$$\nabla_w z(p,w) = (1/p)D^2 f(z(p,w))^{-1}.$$

Note here that, since $D^2 f(z(p,w))$ is negative definite, so is its inverse $D^2 f(z(p,w))^{-1}$ by Theorem M.D.1(iii).

(a) By the chain rule (Section M.A),

$$\frac{d}{dp}[f(z(p,w))] = \nabla f(z(p,w)) \frac{\partial z}{\partial p}(p,w)$$
$$= (-\ 1/p)\nabla f(z(p,w)) \cdot D^2 f(z(p,w))^{-1} \nabla f(z(p,w))$$

Since $D^2 f(z(p,w))^{-1}$ is negative definite, $d[f(z(p,w))]/dp > 0$.

(b) Since $\partial f(z)/\partial z_\ell \geq 0$ for all ℓ and $d[f(z(p,w))]/dp > 0$, as the output price increases, the demand for some input must increase.

(c) Since $\nabla_w z(p,w) = (1/p)D^2 f(z(p,w))^{-1}$, $\partial z(p,w)/\partial p_\ell$ is equal to the ℓth diagonal entry of $(1/p)D^2 f(z(p,w))^{-1}$. Since $D^2 f(z(p,w))^{-1}$ is negative definite, the diagonal entry is negative. Hence so is $\partial z(p,w)/\partial p_\ell$.

5.C.7 [<u>First printing errata</u>: The condition $\partial^2 f(z)/\partial z_\ell \partial z_k < 0$ should be $\partial^2 f(z)/\partial z_\ell \partial z_k > 0$. That is, all inputs are complementary to one another.] As we saw in the answer to Exercise 5.C.6,

$$\frac{\partial z}{\partial w}(p,w) = (1/p)[D^2 f(z(p,w))]^{-1}$$

and

$$\frac{\partial z}{\partial p}(p,w) = -\ (1/p)[D^2 f(z(p,w))]^{-1} \nabla f(z(p,w)).$$

Hence, in order to prove that $\partial z_\ell(p,w)/\partial w_k < 0$ for all $k \neq \ell$ and $\frac{\partial z_\ell}{\partial p}(p,w) >$ 0 for all ℓ, it is sufficient to show that all entries of $[D^2 f(z(p,w))]^{-1}$ are negative. This is an immediate consequence of the celebrated Hawkins-Simon condition, which can be found, for example, in "Convex Structures and Economic Theory" by Hukukane Nikaido. Here, instead of simplemindedly quoting the Hawkins-Simon condition, we shall provide a direct proof that relies on the symmetry of $D^2 f(z(p,w))$ (which is not assumed in the Hawkins-Simon condition).

Write $H = D^2 f(z(p,w))$. To show that all entries of H^{-1} are negative, it is sufficient to prove that, for every $v \in \mathbb{R}^{L-1}$, if $Hv \geq 0$ and $Hv \neq 0$, then $v \ll 0$. In fact, then, for each ℓ, we can choose $v \in \mathbb{R}^{L-1}$ so that Hv is the vector whose ℓth coordinate is equal to one and the other coordinates are equal to zero. Then $H^{-1}(Hv)$ is equal to the ℓth column of H^{-1}. Of course, it is also equal to v, which is claimed to be strictly negative. Thus every column of H^{-1} is strictly negative. Hence all entries of H^{-1} are negative.

For this property, in turn, it is sufficient to prove that for every $v \in \mathbb{R}^{L-1}$, if $Hv \geq 0$, then $v \leq 0$. (That is, weak inequalities suffice.) In fact, if there exists a $v \in \mathbb{R}^{L-1}$ such that $Hv \geq 0$, $Hv \neq 0$, $v \leq 0$, and $v_\ell = 0$ for some ℓ. Then $v \neq 0$ and hence

$$\sum_k (\partial z_\ell(p,w)/\partial w_k) v_k = \sum_{k \neq \ell} (\partial z_\ell(p,w)/\partial w_k) v_k < 0,$$

which contradicts $Hv \geq 0$.

We shall now prove by contradiction that for every $v \in \mathbb{R}^{L-1}$, if $Hv \geq 0$, then $v \leq 0$. Then there exists $v \in \mathbb{R}^{L-1}$ such that $Hv \geq 0$, and $v_\ell > 0$ for some ℓ. By re-ordering the inputs if necessary, we can assume that the first M entries of v are positive and the last L - 1 - M entries are nonpositive Write write $v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where $x_1 \in \mathbb{R}^M$, $x_1 \gg 0$, $x_2 \in \mathbb{R}^{L-1-M}$, and $x_2 \leq 0$. Also, write $H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$, where H_1 is an $M \times M$ matrix, H_2 is $M \times (L - 1 - M)$ matrix whose entries are all positive, H_3 is an $(L - 1 - M) \times M$ matrix whose entries are all positive, and H_4 is an $(L - 1 - M) \times (L - 1 - M)$ matrix. Let $Hv = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, where $y_1 \in \mathbb{R}^M$, $y_1 \geq 0$, $y_2 \in \mathbb{R}^{L-1-M}$, and $y_2 \geq 0$. Then

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} H_1 x_1 + H_2 x_2 \\ H_3 x_1 + H_4 x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Hence $H_1 x_1 = y_1 - H_2 x_2 \geq 0$ because $y_1 \geq 0$, $x_2 \leq 0$, and all entries of H_2 are positive. Thus, by $x_1 >> 0$, we obtain $x_1 \cdot H_1 x_1 \leq 0$, which is a contradiction to the negative definiteness of H, and hence that of H_1.

5.C.8 The cost that AI incurred in month 95 is $2 \cdot 55 + 2 \cdot 40 = 190$, but it could attain the same output level with a lower cost by using the input combination of month 3: $2 \cdot 40 + 2 \cdot 50 = 180$. Thus the problem we will encounter is that, perhaps due to mis-observation and/or some restrictions that AI faced outside the market, the profit-maximizing production plans are not observed to have been used and it is impossible to use those observations in order to recover its technology based on Proposition 5.C.2(iii) or 5.C.1(iii).

5.C.9. To find $\pi(\cdot)$ and $y(\cdot)$ for (a), the first-order condition (5.C.2) is not very useful, because one of the nonnegativity constraint binds. Also, to find $\pi(\cdot)$ and $y(\cdot)$ for (b), it is not even applicable because $f(\cdot)$ is not differentiable. In both cases, however, because of the nature of the production functions, it is quite easy to solve their CMP (which is similar to those in Exercise 5.C.10.), and the cost functions $c(\cdot)$ turn out to be differentiable with respect to output levels q. We can thus apply the first-order condition (5.C.6) (which requires only the differentiability of the cost function with respect to output levels) to find profit maximizing production levels, and hence the profit functions and supply correspondences.

Throughout the answer, the output price is fixed to be equal to one.

(a) $$\pi(w) = \begin{cases} 1/4w_1 & \text{if } w_1 \leq w_2; \\ 1/4w_2 & \text{if } w_1 > w_2. \end{cases}$$

$$y(w) = \begin{cases} \{(-1/4w_1^2,\ 0,\ 1/2w_1)\} & \text{if } w_1 < w_2; \\ \{(-z_1,\ -z_2,\ 1/2w_1): z_1 \geq 0,\ z_2 \geq 0,\ z_1 + z_2 = 1/4w_1^2\} & \text{if } w_1 = w_2; \\ \{(0,\ -1/4w_2^2,\ 1/2w_1)\} & \text{if } w_1 > w_2. \end{cases}$$

(b) $\pi(w) = 1/4(w_1 + w_2)$.

$y(w) = (-1/4(w_1 + w_2)^2, -1/4(w_1 + w_2)^2, 1/2(w_1 + w_2))$.

(c) Note first that this production function exhibits constant returns to scale. Moreover, if $\rho < 1$, then the nonnegativity constraint does not bind. If $\rho = 1$, then this production function gives rise to the same isoquants as that of (a), and hence one of the nonnegativity constraints binds. It is thus easy to apply (5.C.6).

If $\rho < 1$, then

$$\pi(w) = \begin{cases} \infty & \text{if } w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} < 1; \\ 0 & \text{if } w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} \geq 1. \end{cases}$$

$$y(w) = \begin{cases} \emptyset & \text{if } w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} < 1, \\ \{\alpha(-w_1^{1/(\rho-1)}, -w_2^{1/(\rho-1)}, (w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)})^{1/\rho})): \alpha \geq 0\} & \\ & \text{if } w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} = 1; \\ \{0\} & \text{if } w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} > 1. \end{cases}$$

If $\rho = 1$, then

$$\pi(w) = \begin{cases} 0 \text{ if } \mathrm{Min}\{w_1,w_2\} \geq 1 \\ \infty \text{ if } \mathrm{Min}\{w_1,w_2\} < 1 \end{cases}$$

$$y(w) = \begin{cases} \{0\} & \text{if } \mathrm{Min}\{w_1,w_2\} \geq 1; \\ \emptyset & \text{if } \mathrm{Min}\{w_1,w_2\} < 1; \\ \{\alpha(-1, 0, 1): \alpha \geq 0\} & \text{if } 1 = w_1 < w_2; \\ \{\alpha(0, -1, 1): \alpha \geq 0\} & \text{if } w_1 > w_2 = 1. \\ \{\alpha(-z_1, -z_2, 1): \alpha \geq 0, z_1 \geq 0, z_2 \geq 0, z_1 + z_2 = 1\} & \\ & \text{if } w_1 = w_2 = 1; \end{cases}$$

5.C.10

(a) $$c(w,q) = \begin{cases} qw_1 & \text{if } w_1 \leq w_2; \\ qw_2 & \text{if } w_1 > w_2. \end{cases}$$

$$z(w,q) = \begin{cases} (q,0) & \text{if } w_1 < w_2; \\ \{(z_1,z_2) \in \mathbb{R}^2_+ : z_1 + z_2 = q\} & \text{if } w_1 = w_2; \\ (0,q) & \text{if } w_1 > w_2. \end{cases}$$

(b) $c(w,q) = (w_1 + w_2)q$. $z(w,q) = (q,q)$.

(c) $c(w,q) = q(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)})^{(1-1/\rho)}$.

$z(w,q) = q(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)})^{(-1/\rho)}(w_1^{1/(\rho-1)}, w_2^{1/(\rho-1)})$.

5.C.11 Assume that $c(\cdot)$ is twice continuously differentiable. By Proposition 5.C.2(vi), $z(\cdot)$ is continuously differentiable and

$$\partial z_\ell(w,q)/\partial q = (\partial/\partial q)(\partial c(w,q)/\partial w_\ell) = (\partial/\partial w_\ell)(\partial c(w,q)/\partial q).$$

Hence $\partial z_\ell(w,q)/\partial q > 0$ if and only if $(\partial/\partial w_\ell)(\partial C(w,q)/\partial q) > 0$, that is, marginal cost is increasing in w_ℓ.

5.C.12 Suppose first that $y \in Y$ maximizes profit at p, then $(p,\pi(p))\cdot(y, -1) = 0$. Also, for every $\alpha \geq 0$ and $y' \in Y$, $(p,\pi(p))\cdot(\alpha(y', -1)) = \alpha(p\cdot y' - \pi(p)) \leq 0$. Thus $(y', -1)$ maximizes profit in Y' at prices $(p,\pi(p))$.

Conversely, if $y \in Y$ and $(y, -1)$ maximizes profit in Y' at (p,p_{L+1}), then $(p,p_{L+1})\cdot(y, -1) = p\cdot y - p_{L+1} = 0$ by the constant returns to scale. Also, for every $y' \in Y$, $(p,p_{L+1})\cdot(y', -1) = p\cdot y' - p_{L+1} \leq 0$. Hence y maximizes profit in Y at prices p and $\pi(p) = p\cdot y = p_{L+1}$.

5.C.13 Denote the production function of the firm by $f(\cdot)$, then its optimization problem is

$$\text{Max}_{(z_1,z_2)\geq 0}\ pf(z_1,z_2) \quad \text{s.t.} \quad w_1z_1 + w_2z_2 \leq C.$$

This is analogous to the utility maximization problem in Section 3.D and the function $R(\cdot)$ corresponds to the indirect utility function. Hence, analogously to Roy's identity (Proposition 3.G.4), the input demands are obtained as

$$-\frac{1}{\nabla_C R(p,w,C)}\nabla_w R(p,w,C) = (\alpha C/w_1,\ (1-\alpha)C/w_2).$$

5.D.1 We shall use the differentiability of $C(\cdot)$ only at $\bar{q}$. The everywhere differentiability is not necessary. By the definition,

$$AC'(\bar{q}) = \frac{d}{dq}\left[\frac{C(\bar{q})}{\bar{q}}\right] = \frac{C'(\bar{q})\bar{q} - C(\bar{q})}{\bar{q}^2}.$$

Thus, if the average cost is minimized at $q = \bar{q}$, then $AC'(\bar{q}) = 0$ and hence $C'(\bar{q}) = C(\bar{q})/\bar{q} = AC(\bar{q})$.

5.D.2

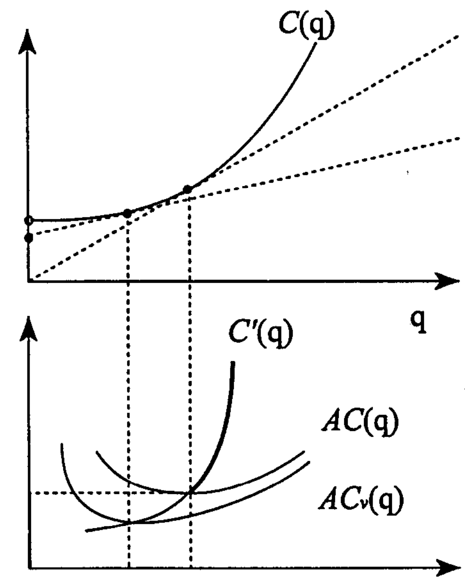


Figure 5.D.2

5.D.3 Let $q(w,p)$ be the profit-maximizing output level when the input prices are w and the output price is p. Let $\bar{w}$ be the initial input prices, $\bar{p}$ be the initial output price, $\bar{z}$ be the initial long-run input demand at $(\bar{w},\bar{p})$, and $\bar{q}$ be the initial long-run output level at $(\bar{w},\bar{p})$. By (5.C.6), $\partial c(\bar{w},q(\bar{w},p))/\partial q = p$ for every p (assuming that $q(\bar{w},p) > 0$). Thus, by differentiating both sides with respect to p and then evaluating at $p = \bar{p}$, we obtain

$$(\partial^2 c(\bar{w}, q(\bar{w},\bar{p}))/\partial q^2)(\partial q(\bar{w},\bar{p})/\partial p) = 1,$$

that is, $\partial q(\bar{w},\bar{p})/\partial p = (\partial c^2(\bar{w},\bar{q})/\partial^2 q)^{-1}$.

On the other hand, define the short-run cost function function $c_s(w,q|\bar{z}_1)$ and the short-run output function $q_s(w,q|\bar{z}_1)$ as suggested in the hint. Just as we did above, we can show that $\partial q_s(\bar{w},\bar{p}|\bar{z}_1)/\partial p = (\partial c_s^2(\bar{w},\bar{q}|\bar{z}_1)/\partial^2 q)^{-1}$ (assuming that $q_s(\bar{w},p|\bar{z}_1) > 0$).

Now, by the definition, $c(\bar{w},q) \le c_s(\bar{w},q|\bar{z}_1)$ for all q and $c(\bar{w},\bar{q}) = c_s(\bar{w},\bar{q}|\bar{z}_1)$. Hence the function $g(q) = c(\bar{w},q) - c_s(\bar{w},q|\bar{z}_1)$ is maximized at $q = \bar{q}$ and, by the second-order necessary condition (see Section M.K), $g''(\bar{q}) \le 0$, that is, $\partial c^2(\bar{w},\bar{q})/\partial^2 q \le \partial c_s^2(\bar{w},\bar{q}|\bar{z}_1)/\partial^2 q$. Thus $\partial q(\bar{w},\bar{p})/\partial p \ge \partial q_s(\bar{w},\bar{p}|\bar{z}_1)/\partial p$.

5.D.4 **(a)** Suppose that $q = \sum_{j=1}^{J} q_j$. By the decreasing average costs (and $C(0) = 0$), $(q_j/q)C(q) \le C(q_j)$. By summing over j, we obtain $C(q) \le \sum_{j=1}^{J} C(q_j)$. Hence there is no way to break up the production of q among multiple firms and lower the cost of production. Hence $C(\cdot)$ is subadditive.

(b) Let $M = 2$ and define $C(q) = \sqrt{\min\{q_1, q_2\}}$, then $C(\cdot)$ exhibits decreasing ray average cost. But let $q_1 = (1,8)$, $q_2 = (8,1)$, and $q = q_1 + q_2 = (9,9)$. Then $C(q_1) = C(q_2) = 1$ and $C(q) = 3$. Hence $C(q) > C(q_1) + C(q_2)$. Hence $C(\cdot)$ is not subadditive.

(c) We shall first prove that if $q = \sum_j q_j$ and $q_j \gg 0$ for every j, then $C(q) \le \sum_j C(q_j)$. In fact, then, for every j, there exists $\gamma_j > 0$ such that $\gamma_j q_j \gg q$. By the increasingness, $C(\gamma_j q_j) > C(q)$. Thus, by the continuity and $C(q) > C(q_j)$, there exists $\alpha_j \in (1, \gamma_j)$ such that $C(\alpha_j q_j) = C(q)$. Define $\beta = \sum_j 1/\alpha_j$, then $\sum_j 1/\beta\alpha_j = 1$ and $\sum_j (1/\beta\alpha_j)(\alpha_j q_j) = (1/\beta)q$. Thus, by the quasiconvexity, $C((1/\beta)q) \le C(q)$. By the increasingness, $\beta \ge 1$. Hence, by the decreasing ray average cost,

$$\sum_j C(q_j) \ge \sum_j (1/\alpha_j) C(\alpha_j q_j) = \sum_j (1/\alpha_j) C(q) = \beta C(q) \ge C(q).$$

For the general case in which some $q_j \ge 0$ need not be strictly positive, apply the above result to the $q_j + \varepsilon e$, where $\varepsilon > 0$ and $e = (1,1,\ldots,1) \in \mathbb{R}^M$, and then take the limit as $\varepsilon \to 0$. The continuity of $C(\cdot)$ then implies that $\sum_{j=1}^J C(q_j) \ge C(q)$.

5.D.5 **(a)** The production function $f(\cdot)$ exhibits increasing returns if and only if $f(\lambda z) \ge \lambda f(z)$ for all z and all $\lambda \ge 1$. Hence, if $z' \ge z > 0$, then

$$(1/z')f(z') = (1/z')f((z'/z)z) \ge (1/z')(z'/z)f(z) = (1/z)f(z).$$

Thus the average product is nondecreasing. The marginal product may however be decreasing on some region of output levels, as the following example shows:

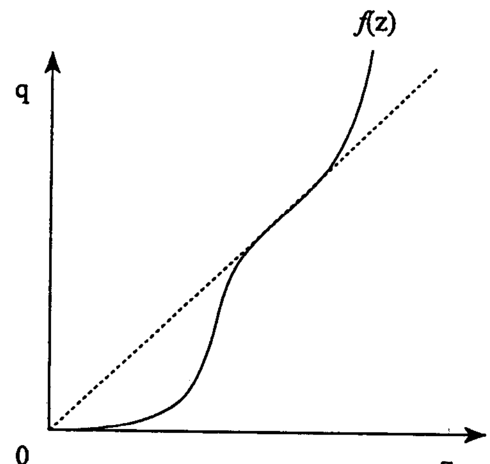


Figure 5.D.5(a)

(b) Mathematically, the consumer's maximization problem is

$$\max_{z \ge 0} u(f(z)) - z.$$

The first-order necessary condition is $u'(f(z))f'(z) = 1$, which can be rewritten as $u'(f(z)) = f'(z)^{-1}$. Since the cost function is given by $z = f^{-1}(q)$, the marginal cost is equal to $f'(z)^{-1}$ and the equality of marginal cost and marginal utility is thus a necessary condition for a maximum.

Economically, the consumer will choose the output level at which the marginal utility of an extra unit of the output is exactly equal to the disutility incurred by giving up the necessary amounts of input to produce it. But the latter is nothing but the marginal cost. Hence the marginal utility is equal to the marginal cost.

(c) This assertion is wrong. As the following figure shows, even if marginal cost and marginal utility are equal at an input level, there may be another input level at which the consumer attains higher utility:

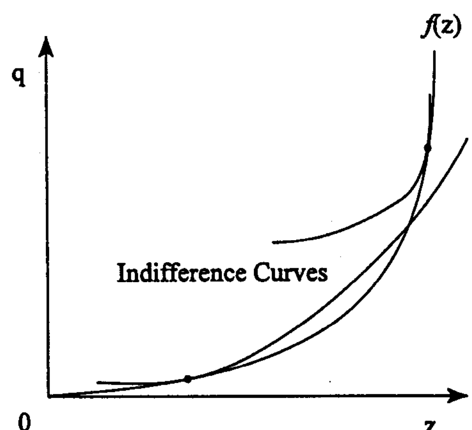


Figure 5.D.5(c)

The reason for suboptimality is that the first-order necessary conditions are not sufficient when the production function exhibits increasing returns (which gives rise to nonconvexity of the feasible set).

5.E.1 By applying Hotelling's lemma (Proposition 5.C.1(vi)) twice, we get

$$y^*(p) = \nabla\pi^*(p) = \nabla(\sum_j \pi_j(p)) = \sum_j \nabla\pi_j(p) = \sum_j y_j(p).$$

5.E.2 This is just a matter of going through the proof of Proposition 5.E.1

and checking that convexity was never used. Its interpretation was given before the statement of the proposition (p. 148). It is a consequence of the very definition of the aggregate production set, that is, it is the sum of the J firms' production sets. It is thus independent of convexity or any other properties of the firms' production sets.

5.E.3 [<u>First printing errata</u>: We should assume that there is a $p^* \gg 0$ and $y^* \in \sum_j Y_j$ such that $p^* \cdot y^* \geq p^* \cdot y$ for every $y \in \sum_j Y_j$. Otherwise, denoting the profit function of $\sum_j Y_j$ by $\pi^*(\cdot)$, the set $\{y \in \mathbb{R}^L: p \cdot y \leq \pi^*(p)$ for all $p \gg 0\}$ may be empty, and all we can obtain is the equality between $\sum_j Y_j$ and $\{y \in \mathbb{R}^L: p \cdot y \leq \pi^*(p)$ for all $p \geq 0\}$. This assumption is also necessary for the validity of Proposition 5.C.1(iii). In fact, its proof should go as follows: It is sufficient to prove that, for every $z \in \mathbb{R}^L \backslash Y$, there exist a $p \gg 0$ such that $p \cdot z > \pi(p)$. Since Y is closed and convex, the separating hyperplane theorem implies the existence of such a nonzero vector p. The free disposal property implies that p must actually be nonnegative. If it is not strictly positive, then take the convex combination $(1 - \varepsilon)p + \varepsilon p^*$ with a sufficiently small $\varepsilon > 0$. Then it is strictly positive, and satisfies $((1 - \varepsilon)p + \varepsilon p^*) \cdot z > \pi((1 - \varepsilon)p + \varepsilon p^*)$ by the upper semicontinuity of $\pi(\cdot)$, which is implied by its convexity.] Since each Y_j is convex and satisfies the free disposal property, $\sum_j Y_j$ is also convex and satisfies the free disposal property. Since it is also assumed to be closed, Proposition 5.C.1(iii) implies that

$$\sum_j Y_j = \{y \in \mathbb{R}^L: p \cdot y \leq \pi^*(p) \text{ for all } p \gg 0\}.$$

But here, by Proposition 5.E.1, $\pi^*(p) = \sum_j \pi_j(p)$ and hence

$$\sum_j Y_j = \{y \in \mathbb{R}^L: p \cdot y \leq \sum_j \pi_j(p) \text{ for all } p \gg 0\}.$$

5.E.4 (a) Denote by $y_z(w) \subset \mathbb{R}^3$ the set of the supplies of the technology with characteristics $z = (z_1, z_2)$ at input prices $w = (w_1, w_2)$, then

$$y_z(w) = \begin{cases} \{(-z_1, -z_2, 1)\} & \text{if } w_1z_1 + w_2z_2 < 1, \\ \{(-\alpha z_1, -\alpha z_2, \alpha): \alpha \in [0,1]\} & \text{if } w_1z_1 + w_2z_2 = 1, \\ \{0\} & \text{if } w_1z_1 + w_2z_2 > 1. \end{cases}$$

The area of the characteristics z for which the output may be one is depicted in the following picture.

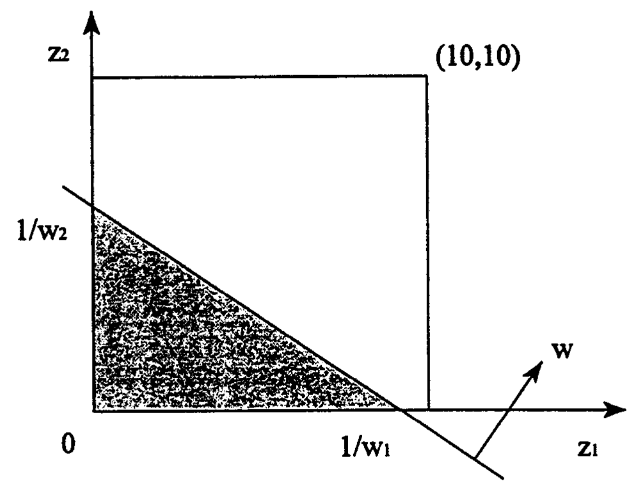


Figure 5.E.4

(b) [<u>First printing errata</u>: The phrase "More generally" should be deleted.] Denote the profit of the technology with characteristics $z = (z_1, z_2)$ at input prices $w = (w_1, w_2)$ by $\pi_z(w)$, then

$$\pi_z(w) = \begin{cases} 1 - w_1z_1 - w_2z_2 & \text{if } w_1z_1 + w_2z_2 \le 1, \\ 0 & \text{if } w_1z_1 + w_2z_2 > 1. \end{cases}$$

Thus, the aggregate (or, rigorously, average) profit is calculated by taking the integral of $1 - w_1z_1 - w_2z_2$ on the area $\{z \in [0,10] \times [0,10]: w_1z_1 + w_2z_2 \le 1\}$, which is depicted on the above figure. Thus the aggregate profit is

$$\pi(w) = \int_0^{1/w_1}\int_0^{(1-w_1z_1)/w_2} (1 - w_1z_1 - w_2z_2)dz_2dz_1 = 1/600w_1w_2.$$

(c) The aggregate input demand can also be obtained by integrating the input demands of the individual firms, but the following point is noteworthy: If a firm has characteristic $z = (z_1,z_2)$ with $w_1z_1 + w_2z_2 = 1$, then it has multiple input demands at input prices $w = (w_1,w_2)$. But those firms constitute only a negligible portion in the whole population. Hence it is harmless to assume that such a firm has input demand $z = (z_1,z_2)$ (in absolute values). Hence the aggregate demands are

$$\int_0^{1/w_1}\int_0^{(1-w_1z_1)/w_2} z_1 \frac{1}{100}\, dz_2 dz_1 = \frac{1}{600w_1^2w_2},$$

$$\int_0^{1/w_1}\int_0^{(1-w_1z_1)/w_2} z_2 \frac{1}{100}\, dz_2 dz_1 = \frac{1}{600w_1w_2^2}.$$

It is easy to check that these aggregate input demand functions can also be obtained by applying Hotelling's lemma to the aggregate profit function, which was obtained in (b).

(d) We need to find an aggregate production function whose input demand function is the same as the aggregate input demand function in (c). Denote such a production function by $f(\cdot)$, then the first order-conditions for profit maximization are $\partial f(z_1,z_2)/\partial z_1 = w_1$ and $\partial f(z_1,z_2)/\partial z_2 = w_2$. These expressions, when evaluated at the inputs demanded, must hold for all w. Thus

$$\frac{\partial f}{\partial z_1}\left(\frac{1}{600w_1^2w_2}, \frac{1}{600w_1w_2^2}\right) = w_1, \quad \frac{\partial f}{\partial z_2}\left(\frac{1}{600w_1^2w_2}, \frac{1}{600w_1w_2^2}\right) = w_2.$$

Let

$$z_1 = 1/600w_1^2w_2 \text{ and } z_2 = 1/600w_1w_2^2,$$

then

$$w_1 = (z_2/600z_1^2)^{1/3} \text{ and } w_2 = (z_1/600z_2^2)^{1/3}.$$

Thus

$$\frac{\partial f}{\partial z_1}(z_1,z_2) = (z_2/600z_1^2)^{1/3} \text{ and } \frac{\partial f}{\partial z_2}(z_1,z_2) = (z_1/600z_2^2)^{1/3}.$$

Therefore, $f(z_1,z_2) = 3(z_1z_2/600)^{1/3}$. The aggregate production function is a Cobb-Douglas one exhibiting decreasing returns to scale.

5.E.5 **(a)** Plant j's marginal cost is $MC_j(q_j) = \alpha + 2\beta_j q_j$. Since $\beta_j > 0$ for every j, the first-order necessary and sufficient conditions for cost minimization are that $\sum_j q_j = q$ and $MC_j(q_j) = MC_{j'}(q_{j'})$ for all j and j'. From these, we obtain $q_j = (q/\beta_j)/(\sum_h 1/\beta_h)$.

(b) (c) In both cases, it is cost-minimizing to concentrate on plants with the smallest $\beta_j < 0$, because the average cost is decreasing at the highest rate at such plants.

5.F.1 The production plan y in Figure 5.F.1(b) is not efficient but it maximizes profit for p = (0,1).

5.G.1 Throughout the answer, we fix the price of the input at one and denote the price of the output by p. Suppose that there are I consumer-owners, indexed by i = 1,...,I. Denote their shares by $\theta_i > 0$. Of course, $\sum_i \theta_i = 1$. Since they have quasilinear utility functions, by Exercise 3.D.4(b), their indirect utility functions can be written as $v_i(p,w_i) = w_i + \phi_i(p)$. Note that the demand function $x_i(\cdot)$ for the output of consumer i does not depend on the wealth and satisfies $x_i(p) = -\phi_i'(p)$ by Roy's identity.

(a) When the input is z, the utility level of consumer i is

$$\theta_i(p(z)f(z) - z) + \phi_i(p(z)).$$

(Here we are assuming that the consumers have no source of wealth other than their shareholdings. But this does not affect our results below, because their demands for the output does not depend on their wealth levels. Thus, if an input level z maximizes his utility level, then it satisfies the following first-order condition:

$$\theta_i(p'(z)f(z) + p(z)f'(z) - 1) + \phi_i'(p(z))p'(z) = 0.$$

If an input level z is unanimously agreed, then this first-order condition must be satisfied at some z for all i. By taking the summation of the condition over i and using $x_i(p(z)) = -\phi_i'(p(z))$, we obtain

$$(p'(z)f(z) + p(z)f'(z) - 1) - \sum_i x_i(p(z))p'(z) = 0.$$

But, since $f(z) = \sum_i x_i(p(z))$, this implies $p(z)f'(z) - 1 = 0$. Plugging this into the first-order condition, we obtain

$$\theta_i p'(z)f(z) - x_i(p(z))p'(z) = 0.$$

Thus $\theta_i = x_i(p(z))/f(z)$.

(b) We know from (a) that, if ownership share are identical, then, in order for consumer-owners to unanimously agree on a production plan, it is necessary that they all consume the same amount of the output. But if their tastes are different for the output, then their consumption levels will be different. Hence they will instruct managers to carry out different output levels.

(c) If preferences and ownership shares are identical, then the first-order conditions are also identical and hence the consumer-owners unanimously agree on an input level. We showed in (a) that, a necessary condition for the unanimous agreement is that $p(z) = 1/f'(z)$. The right-hand side is the inverse of the marginal return, and hence equal to the marginal. This is nothing but profit maximization with respect to input, when the output price $p(z)$ is taken as given.

5.AA.1 From the unit isoquant,

$$z(w,1) = \begin{cases} (2,1) & \text{if } w_1 < w_2, \\ \{\lambda(2,1) + (1-\lambda)(1,2) \in \mathbb{R}: \lambda \in [0,1]\} & \text{if } w_1 = w_2, \\ (1,2) & \text{if } w_1 > w_2. \end{cases}$$

Thus

$$c(w,1) = \begin{cases} 2w_1 + w_2 & \text{if } w_1 \leq w_2, \\ w_1 + 2w_2 & \text{if } w_1 > w_2. \end{cases}$$

This is differentiable at $w = (w_1,w_2)$ if and only if $w_1 \neq w_2$. Moreover, $\nabla c(w,1) = z(w,1)$ at $w = (w_1,w_2)$ with $w_1 \neq w_2$.

5.AA.2 **(a)** We shall first prove that if $\beta \in \mathbb{R}^{L-1}$, $(I - A)\beta \geq 0$, and $(I - A)\beta \neq 0$, then $b\cdot\beta > 0$, and that if $\beta \in \mathbb{R}^{L-1}$ and $(I - A)\beta = 0$, then $b\cdot\beta = 0$. In fact, in the proof of Proposition 5.AA.1, we showed that since A is productive, the inverse matrix $(I - A)^{-1}$ exists and all its entries are nonnegative. Thus, if $\beta \in \mathbb{R}^{L-1}$, $(I - A)\beta \geq 0$, and $(I - A)\beta \neq 0$, then $\beta = (I - A)^{-1}((I - A)\beta) \geq 0$ and $\beta \neq 0$. Since $b \gg 0$, this implies $b\cdot\beta > 0$. If $\beta \in \mathbb{R}^{L-1}$ and $(I - A)\beta = 0$, then $\beta = 0$ and hence $b\cdot\beta = 0$.

To derive efficiency from the above result, let $\alpha \in \mathbb{R}^{L-1}_+$ and $\alpha' \in \mathbb{R}^{L-1}_+$. If $(I - A)\alpha \geq (I - A)\alpha'$ and $(I - A)\alpha \neq (I - A)\alpha'$, then $(I - A)(\alpha - \alpha') \geq 0$ and $(I - A)(\alpha - \alpha') \neq 0$. Hence $b\cdot(\alpha - \alpha') > 0$, or $b\cdot\alpha > b\cdot\alpha'$. If $(I - A)\alpha = (I - A)\alpha'$, then $(I - A)(\alpha - \alpha') = 0$. Hence $b\cdot(\alpha - \alpha') = 0$, or $b\cdot\alpha = b\cdot\alpha'$. In any case, it is impossible that $\begin{bmatrix} I - A \\ -b \end{bmatrix}\alpha \geq \begin{bmatrix} I - A \\ -b \end{bmatrix}\alpha'$ and $\begin{bmatrix} I - A \\ -b \end{bmatrix}\alpha \neq \begin{bmatrix} I - A \\ -b \end{bmatrix}\alpha'$. Efficiency is thus established.

(b) By (a) and Proposition 5.F.2, any production plan with $\alpha \gg 0$ is profit-maximizing at some price vector. To establish its uniqueness, let $p \in \mathbb{R}^{L-1}_+$ be a supporting price vector. By $\alpha \gg 0$ and the zero-profit condition for activities being actually used, we must have $p\cdot(I - A) = b$, that is, $p = ((I - A)^{-1})^T b$ (where b is now a column vector). This implies the uniqueness and the strict positivity of p, because all entries of $(I - A)^{-1}$ are nonnegative, all its diagonal entries of are positive, and $b \gg 0$.

(c) For each ℓ, denote by e_ℓ the vector in $\mathbb{R}^{L-1}$ whose ℓth component is one and the other components are zero. As we saw in the remark following Proposition 5.AA.1, the total amounts of the producible goods necessary to realize a net output vector $e_\ell \in \mathbb{R}^{L-1}_+$ is equal to $(\sum_{n=0}^{\infty} A^n)e_\ell = (I - A)^{-1}e_\ell$. Hence the total amount of labor embodied in these necessary amounts of the producible goods equals $b\cdot(\sum_{n=0}^{\infty} A^n)e_\ell = b\cdot(I - A)^{-1}e_\ell = p_\ell$. Thus the price (row) vector $p = b\cdot(I - A)^{-1}$ can be interpreted as the amounts of primary factor directly or indirectly embodied in the production of one unit of each producible good.

(d) Let $\begin{bmatrix} I - A' \\ - b' \end{bmatrix} \in \mathbb{R}^{L\times(L-1)}$ be any alternative choice of activities that is productive. By the productivity, the inverse matrices $(I - A)^{-1}$ and $(I - A')^{-1}$ exist and are nonnegative. So denote them by $C = [c_1 \ldots c_{L-1}]$ and $C' = [c'_1 \ldots c'_{L-1}]$, where the c_ℓ and c'_ℓ ($\ell = 1,\ldots, L - 1$) are $(L - 1)$-dimensional column vectors. Then $(I - A)c_\ell = e_\ell$ and $(I - A')c'_\ell = e_\ell$. Now, for each $\varepsilon > 0$, define $d_\ell(\varepsilon) = c_\ell + \varepsilon(\sum_{k\neq\ell}c_k)$ and $d'_\ell(\varepsilon) = c'_\ell + \varepsilon(\sum_{k\neq\ell}c'_k)$. Then $d_\ell(\varepsilon) \to c_\ell$ and $d'_\ell(\varepsilon) \to c'_\ell$ as $\varepsilon \to 0$. Moreover, $d_\ell(\varepsilon) \gg 0$, $d'_\ell(\varepsilon) \gg 0$, and

$$(I - A)d_\ell(\varepsilon) = (I - A')d'_\ell(\varepsilon) = e_\ell + \varepsilon(\textstyle\sum_{k\neq\ell}e_k) \gg 0.$$

By (a), $\begin{bmatrix} I - A \\ - b \end{bmatrix}d_\ell(\varepsilon)$ is efficient. By this strict positivity and the assumption that the activities $\begin{bmatrix} I - A \\ - b \end{bmatrix}$ have been singled out by the nonsubstitution theorem, we must have $b\cdot d_\ell(\varepsilon) \le b'\cdot d'_\ell(\varepsilon)$. Taking the limit as $\varepsilon \to 0$, we obtain $b\cdot c_\ell \le b'\cdot c'_\ell$. Thus $b\cdot(I - A)^{-1} \le b'\cdot(I - A')^{-1}$. The assertion now follows from (c).

5.AA.3 **(a)** Denote the activity levels by α_1 and α_2. The resource constraint for labor is $\alpha_1 + 2\alpha_2 = 10$, or $\alpha_1/10 + \alpha_2/5 = 1$. Since the production level is given by $\begin{bmatrix} 10 & -5 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} \alpha_1/10 \\ \alpha_2/5 \end{bmatrix}$, the production possibility frontier is as follows:

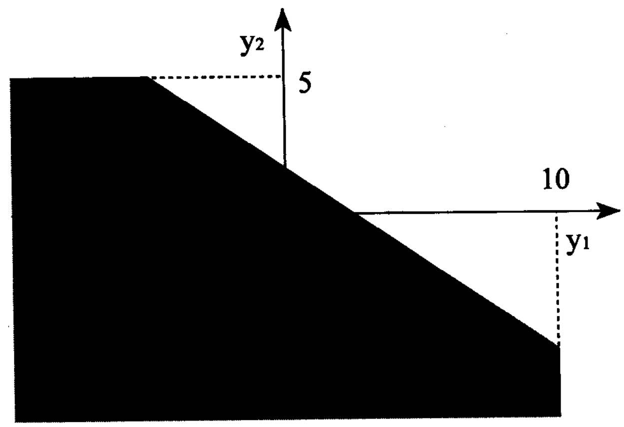


Figure 5.AA.3(a)

(b) By Exercise 5.AA.2(c), the equilibrium price vector is $b\cdot(I - A)^{-1}$ = (4,6).

(c) The amount of labor embodied in each commodity equals its price, as shown in Exercise 5.AA.2(c).

(d) The locus of amounts of good 1 and labor necessary to produce one unit of good 2 is equal to

$$\{\lambda(1,2) + (1 - \lambda)(1/2,\beta): \lambda \in [0,1]\} - \mathbb{R}^2_+,$$

assuming free disposal. It is represented in the following figure:

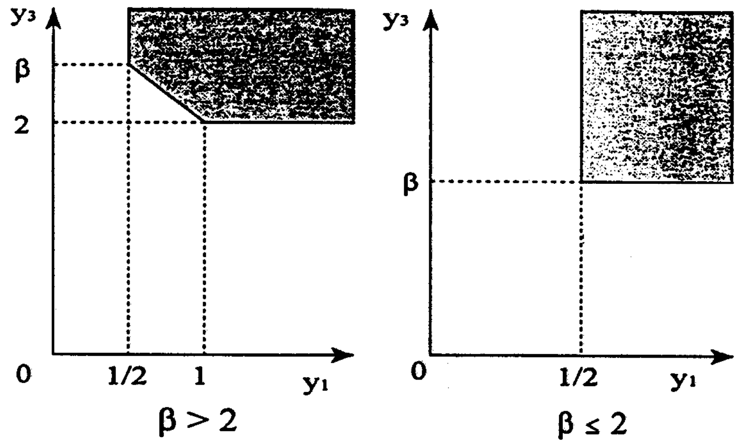


Figure 5.AA.3(d)

(e) In the context of (d), the nonsubstitution theorem says that it is possible to choose one of the two techniques to produce good 2 (or a combination of the two with a fixed proportion) in such a way that any efficient production plan with positive net outputs of the two producible goods can be attained by using the technique chosen for good 2.

We could determine which of the two techniques (or their mixtures) is efficient by actually plotting the frontier of the feasible output combinations from one unit of labor. In the following, however, we shall identify an efficient technique based on Exercise 5.AA.2(d). So let $A' = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ and $b' = \begin{bmatrix} 1 \\ \beta \end{bmatrix}$. For each $\lambda \in [0,1]$, define $A(\lambda) = (1 - \lambda)A + \lambda A$, $b(\lambda) = (1 - \lambda)b + \lambda b'$, and $p(\lambda) = b(\lambda)\cdot(I - A(\lambda))^{-1}$ (where $p(\lambda)$ is a row vector). According to Exercise 5.AA.2(d), if $\lambda^* \in [0,1]$ and the convex combination of the first and the second technique with weight $1 - \lambda^*$ and λ^* is efficient, then $p(\lambda) \geq p(\lambda^*)$ for every $\lambda \in [0,1]$. Hence the switch of efficient techniques occurs precisely when the value of λ^* switches as β varies. We shall now find a value of β at which the value of λ^* switches.

Just as in (b), we can calculate

$$p(\lambda) = \frac{2}{\lambda + 2}\begin{bmatrix} (\beta - 2)\lambda + 4 \\ (2\beta - 5)\lambda + 6 \end{bmatrix} \text{ and } Dp(\lambda) = \frac{2\beta - 8}{(\lambda + 2)^2}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Hence: if $\beta < 4$, then $\lambda^* = 0$; if $\beta > 4$, then $\lambda^* = 1$; and if $\beta = 4$, then λ^* can be any value in [0,1]. Thus the switching occurs at $\beta = 4$. More precisely: if $\beta < 4$, then it is efficient to continue using the first technique; if $\beta > 4$, then it is efficient to switch to the second technique; and if $\beta = 4$, every mixture of the two technique is efficient.

5.AA.4 **(a)** Since $y_2 = 3a_1 + a_4$, $y_3 = 3(a_2 + a_3 + a_4)$, and $y_4 = 4(a_2 + a_3)$, these three vectors are in the production set. But y_1 and y_5 are not. To see this, suppose that $y_1 \le \sum_j \alpha_j a_j$ with $\alpha_j \ge 0$. According to good 2, $\alpha_1 = \alpha_2 = 0$. By $\alpha_2 = 0$, according to good 3, $\alpha_3 = 0$. But there is no $\alpha_4 \ge 0$ for which $y_1 \le \alpha_4 a_4$. Suppose next that $y_5 \le \sum_j \alpha_j a_j$ with $\alpha_j \ge 0$. According to good 1, $\alpha_1 = \alpha_4 = 0$. By $\alpha_4 = 0$, according to good 4, $\alpha_3 = 0$. But there is no $\alpha_2 \ge 0$ for which $y_5 \le \alpha_2 a_2$.

(b) If p = (1,3,3,2), then $p \cdot a_j \le 0$ for all j and $p \cdot y = 0$. Hence y maximizes profit at p. By Proposition 5.F.1, y is efficient.

(c) Since $y = a_1$, y is feasible. But, since $a_2 + a_3 + a_4 = (2, -1, 0, 0)$ is feasible, y cannot be efficient. (Note that $a_2 + a_3 + a_4$ represents a round-about production of good 1 out of good 2.)

5.AA.5 [<u>First printing errata</u>: The last elementary activity $a_8 = (-2, -4, 5, 2)$ should be $a_3 = (-2, -4, 5, -2)$.]

(a) The set Y is defined as $\{\sum_m \alpha_m a_m \in \mathbb{R}^4 : \alpha_m \ge 0$ for each m$\}$. Let $\lambda \in [0,1]$, $y = \sum_m \alpha_m a_m \in Y$, and $y' = \sum_m \alpha'_m a_m \in Y$. Then

$$\lambda y + (1 - \lambda)y' = \sum_m (\lambda\alpha_m + (1 - \lambda)\alpha'_m)a_m.$$

Since $\lambda\alpha_m + (1 - \lambda)\alpha'_m \geq 0$ for every m, $\lambda y + (1 - \lambda)y' \in Y$. Thus Y is convex.

(b) Since all the activities use commodities 1 and 2 as an input, in order to produce any commodity in positive quantity, it is necessary to use commodities 1 and 2 as an inputs. The no-free-lunch property thus follows.

(c) Note that it is impossible to dispose of one of commodities 3 and 4 without increasing the output of the other, and that it is impossible to dispose of any one of commodities 1 and 2 without disposing the other. Hence Y does not satisfy the free-disposal property and it is necessary to add the four disposal activities to the given elementary activities in order for the free-disposal property to be satisfied.

(d) Note that $3a_1 \geq a_5$, $3a_1 \neq a_5$, $a_2 \geq a_4$, $a_2 \neq a_4$, $2a_3 \geq a_8$, $2a_3 \neq a_8$, $a_7 \geq 2a_6$, and $a_7 \neq 2a_6$. Hence a_5, a_4, a_8 and a_6 are not efficient.

(e) We can check that $(4/3)a_3 + (5/24)a_7 \geq a_1$, $(4/3)a_3 + (5/24)a_7 \neq a_1$, $a_3 + (1/2)a_7 \geq a_2$, and $a_3 + (1/2)a_7 \neq a_2$. Hence a_1 and a_2 are not efficient.

(f) We shall prove that the set of the efficient production vectors is equal to $\{\alpha_3 a_3 + \alpha_7 a_7: \alpha_3 \geq 0, \alpha_7 \geq 0\}$. We have shown that every efficient production vector belongs to this set. Conversely, we can check that the production vectors in this set cannot be dominated by each other. Hence they are all efficient and the set of the efficient production vectors is equal to $\{\alpha_3 a_3 + \alpha_7 a_7: \alpha_3 \geq 0, \alpha_7 \geq 0\}$.

(g) Since $\alpha_3 a_3 + \alpha_7 a_7 = (-\alpha_3 - 8\alpha_7, -2\alpha_3 - 5\alpha_7, 3\alpha_3, -\alpha_3 + 10\alpha_7)$, the problem of maximizing the net production of the third commodity is as follows:

$$\text{Max}_{(\alpha_3,\alpha_7)}\ 3\alpha_3$$

$$\begin{aligned} \text{s.t.} \quad \alpha_3 + 8\alpha_7 &\le 480, \\ 2\alpha_3 + 5\alpha_7 &\le 300, \\ \alpha_3 - 10\alpha_7 &\le 0, \\ \alpha_3 &\ge 0, \\ \alpha_7 &\ge 0. \end{aligned}$$

(h) The feasible set is shaded in the (α_3,α_7)-space below.

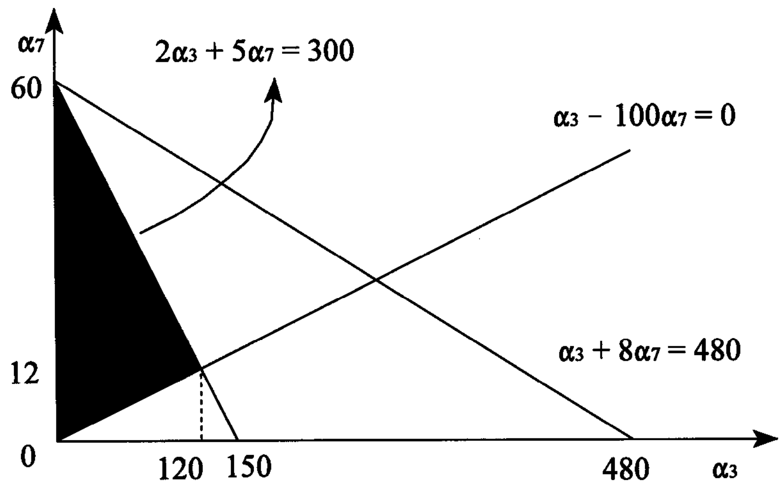


Figure 5.AA.5(h)

The solution to this problem satisfies $2\alpha_3 + 5\alpha_7 = 300$, $\alpha_3 = 10\alpha_7$. Thus $(\alpha_3,\alpha_7) = (120,12)$.

CHAPTER 6

6.B.1 Suppose first that $L \succ L'$. A first application of the independence axiom (in the "only-if" direction in Definition 6.B.4) yields

$$\alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''.$$

If these two compound lotteries were indifferent, then a second application of the independence axiom (in the "if" direction) would yield $L' \succsim L$, which contradicts $L \succ L'$. We must thus have

$$\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L''.$$

Suppose conversely that $\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L''$, then, by the independence axiom, $L \succsim L'$. If these two simple lotteries were indifferent, then the independence axiom would imply

$$\alpha L' + (1 - \alpha)L'' \succsim \alpha L + (1 - \alpha)L'',$$

a contradiction. We must thus have $L \succ L'$.

Suppose next that $L \sim L'$, then $L \succsim L'$ and $L' \succsim L$. Hence by applying the independence axiom twice (in the "only if" direction), we obtain

$$\alpha L + (1 - \alpha)L'' \sim \alpha L' + (1 - \alpha)L''.$$

Conversely, we can show that if $\alpha L + (1 - \alpha)L'' \sim \alpha L' + (1 - \alpha)L''$, then $L \sim L'$.

For the last part of the exercise, suppose that $L \succ L'$ and $L'' \succ L'''$, then, by the independence axiom and the first assertion of this exercise,

$$\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L''$$

and

$$\alpha L' + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L'''.$$

Thus, by the transitivity of $\succ$ (Proposition 1.B.1(i)),

$$\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L'''.$$

6.B.2 Assume that the preference relation $\succsim$ is represented by an v.N-M expected utility function $U(L) = \sum_n u_n p_n$ for every $L = (p_1,\ldots,p_N) \in \mathcal{L}$. Let $L = (p_1,\ldots,p_N) \in \mathcal{L}$, $L' = (p'_1,\ldots,p'_N) \in \mathcal{L}$, $L'' = (p''_1,\ldots,p''_N) \in \mathcal{L}$, and $\alpha \in (0,1)$. Then $L \succsim L'$ if and only if $\sum_n u_n p_n \geq \sum_n u_n p'_n$. This inequality is equivalent to

$$\alpha(\textstyle\sum_n u_n p_n) + (1 - \alpha)(\sum_n u_n p''_n) \geq \alpha(\sum_n u_n p'_n) + (1 - \alpha)\ (\sum_n u_n p''_n).$$

This latter inequality holds if and only if $\alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''$. Hence $L \succsim L'$ if and only if $\alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''$. Thus the independence axiom holds.

6.B.3 Since the set C of outcomes is finite, there are best and worst outcomes in C. Let $\bar{L}$ be the lottery that yields a particular best outcome with probability one and $\underline{L}$ be the lottery that yields a particular worst outcome with probability one. We shall now prove that $\bar{L} \succsim L \succsim \underline{L}$ for every $L \in \mathcal{L}$ by applying the following lemma:

<u>Lemma</u>: Let $L_0, L_1, \ldots, L_K$ be $(1 + K)$ lotteries and $(\alpha_1,\ldots,\alpha_K) \geq 0$ be probabilities with $\sum_{k=1}^{K}\alpha_k = 1$. If $L_k \succsim L_0$ for every k, then $\sum_{k=1}^{K}\alpha_k L_k \succsim L_0$. If $L_0 \succsim L_k$ for every k, then $L_0 \succsim \sum_{k=1}^{K}\alpha_k L_k$.

<u>Proof of Lemma</u>: We shall prove this lemma by induction on K. If K = 1, there is nothing to prove. So let K > 1 and suppose that the lemma is true for K - 1. Assume that $L_k \succsim L_0$ for every k. By the definition of a compound lottery,

$$\sum_{k=1}^{K}\alpha_k L_k = (1 - \alpha_K)\sum_{k=1}^{K-1}\frac{\alpha_k}{1 - \alpha_K}L_k + \alpha_K L_K.$$

By the induction hypothesis, $\sum_{k=1}^{K-1}\frac{\alpha_k}{1 - \alpha_K}L_k \succsim L_0$. Hence, as our first application of the independence axiom, we obtain

$$(1 - \alpha_K)\sum_{k=1}^{K-1}\frac{\alpha_k}{1 - \alpha_K}L_k + \alpha_K L_K \succsim (1 - \alpha_K)L_0 + \alpha_K L_K$$

Applying the axiom once again, we obtain

$$(1 - \alpha_K)L_0 + \alpha_K L_K \succsim (1 - \alpha_K)L_0 + \alpha_K L_0 = L_0.$$

Hence, by the transitivity, $\sum_{k=1}^{K} \alpha_k L_k \succsim L_0$. The first statement is thus verified. The case of $L_0 \succsim L_k$ can similarly be verified.

Now, for each n, let L^n be the lottery that yields outcome n with probability one. Then $\bar{L} \succsim L^n$ because both of them can be identified with sure outcomes. Let $L = (p_1,\ldots,p_N)$ be any lottery, then $L = \sum_n p_n L^n$. Thus, $\bar{L} \succsim L$ the above lemma. The same argument can be used to prove that $L \succsim \underline{L}$.

6.B.4 [First printing errata: On the the 11th and the 12th line of the exercise, the phrase "the lottery of B with probability q and D with probability 1 - q" should be "the lottery of A with probability q and D with probability 1 - q". Also, in the description of Criterion 2, the phrase "an unnecessary evacuation in 5%" should be "an unnecessary evacuation in 15%".]

(a) We can choose an assign utility levels (u_A,u_B,u_C,u_D) so that $u_A = 1$ and $u_D = 0$ as a normalization (Proposition 6.B.2). Then $u_B = p\cdot 1 + (1 - p)\cdot 0 = p$ and $u_C = q\cdot 1 + (1 - q)\cdot 0 = q$.

(b) The probability distribution under Criterion 1 is

$$(p_A,p_B,p_C,p_D) = (0.891,\ 0.099,\ 0.009,\ 0.001).$$

The probability distribution under Criterion 2 is

$$(p_A,p_B,p_C,p_D) = (0.8415,\ 0.1485,\ 0.0095,\ 0.0005).$$

The expected utility under Criterion 1 is thus $0.891 + 0.099p + 0.009q$. The expected utility under Criterion 2 is thus $0.8415 + 0.1485p + 0.0095q$. Hence the agency would prefer Criterion 1 if and only if $99 > 99p + q$, and it would prefer Criterion 2 if and only if $99 < 99p + q$.

6.B.5 **(a)** This follows from Exercise 6.B.1.

(b) The equivalence of the betweenness axiom and straight indifference curves can be established in the same way as in the part of Section 6.B on pp. 175-176 that explains how the independence axiom implies straight indifference curves. (Note that the argument there does not use the fully fledged independence axiom; as it is concerned with two indifferent lotteries, the betweenness axiom suffices.) Those straight lines need not be parallel, because the betweenness axiom imposes restrictions only on straight indifferent curves and nothing on the relative positions of different indifference lines. In fact, the argument for Figure 6.B.5(c) is not applicable to the betweenness axiom.

(c) Any preference represented by straight, but not parallel indifference curves satisfies the betweenness axiom but does not satisfy the independence axiom. Hence the former is weaker than the latter.

(d) Here is an example of a preference relation and its indifference map that satisfies the betweenness axiom and yields the choice of the Allais paradox.

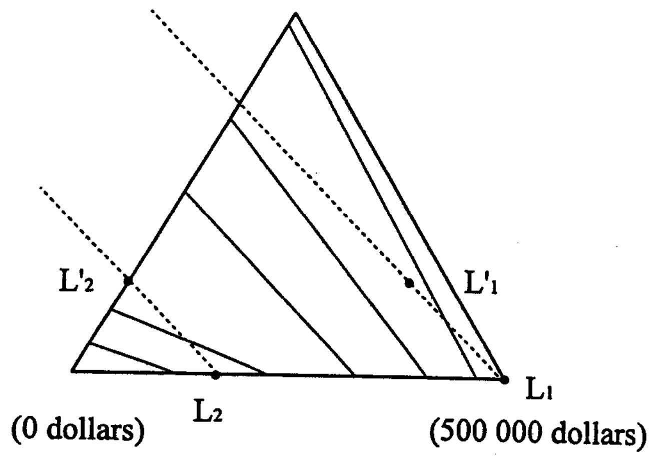


Figure 6.B.5(d)

6.B.6 Define $C = \{(u_1(a),\ldots,u_N(a)) \in \mathbb{R}^N\colon a \in A\}$, then

$$U(p) = \text{Max}\{p\cdot c \in \mathbb{R}\colon c \in C\} = -\text{Min}\{p\cdot c \in \mathbb{R}\colon c \in -C\}.$$

Hence $U(\cdot)$ is equal to $-\mu_{-C}(\cdot)$, the support function (Section 3.F) of $-C$ multiplied by -1, where the domain of the support function is restricted to the simplex $\{p \in \mathbb{R}^N_+\colon \sum_n p_n = 1\}$. Since any support function is concave, $U(\cdot)$ is convex. (A more direct proof is possible, which is essentially the same as the proof of concavity of support functions in Section 3.F.)

As an example of a nonlinear Bernoulli utility function, consider A = B = {1,2} and define $u_1(1) = u_2(2) = 1$ and $u_2(1) = u_1(2) = 0$. Let $L = (p_1,p_2)$, then $U(L) = \text{Max}\ \{p_1,p_2\}$. (This is essentially the same as Example 6.B.4.)

6.B.7 Since the individual prefers L to L' and is indifferent between L and x_L and between L' and $x_{L'}$, by Proposition 1.B.1(iii), he prefers x_L to $x_{L'}$. By the monotonicity, this this is equivalent to $x_L > x_{L'}$.

6.C.1 If $\alpha = D > 0$ (complete insurance), then

$$- q(1 - \pi)u'(w - \alpha q) + \pi(1 - q)u'(w - D + \alpha(1 - q))$$

$$= - q(1 - \pi)u'(w - Dq) + \pi(1 - q)u'(w - Dq)$$

$$= u'(w - Dq)(\pi(1 - q) - q(1 - \pi)) < 0$$

$$= u'(w - Dq)(\pi - q) < 0$$

by $q > \pi$. Thus the first-order condition is not satisfied at $\alpha = D$. Hence the individual will not insure completely.

6.C.2 **(a)** Let $F(\cdot)$ be a distribution function, then

$$\int u(x)dF(x) = \int(\beta x^2 + \gamma x)dF(x) = \beta\int x^2 dF(x) + \gamma\int x dF(x)$$

$$= \beta(\text{mean of } F)^2 + \beta(\text{variance of } F) + \gamma(\text{mean of } F).$$

(b) We prove by contradiction that $U(\cdot)$ is not compatible with any Bernoulli utility function. So suppose that there is a Bernoulli utility function $u(\cdot)$ such that $U(F) = \int u(x)dF(x)$ for every distribution function $F(\cdot)$. Let x and y be two amounts of money, $G(\cdot)$ be the distribution that puts probability one at x, and $H(\cdot)$ be the distribution that puts probability one at y. Then

$$u(x) = U(G) = (\text{mean of } G) - (\text{variance of } G) = x - 0 = x,$$

$$u(y) = U(H) = (\text{mean of } H) - (\text{variance of } H) = y - 0 = y.$$

Thus, $x \geq y$ if and only if $u(x) \geq u(y)$. Hence $u(\cdot)$ is strictly monotone. Now let $F_0(\cdot)$ be the distribution that puts probability one on 0 and $F(\cdot)$ be the distribution that puts probability 1/2 on 0 and on $4/r > 0$. Since the mean and the variance of $F_0(\cdot)$ are zero, $U(F_0) = 0$. The strict monotonicity of $u(\cdot)$ thus implies that $U(F) > 0$. However, the mean of $F(\cdot)$ is $2/r$ and the variance is $4/r^2$. Hence $U(F) = 2/r - r(4/r^2) = - 2/r < 0$, which is a contradiction. Hence $U(\cdot)$ is not compatible with any Bernoulli utility function.

An example of two lotteries with the property requested in the exercise was given in the above proof of incompatibility. (Note that if all we need to show were the incompatibility of $U(\cdot)$ and any Bernoulli utility function, the equality $u(x) = x$ obtained above would be sufficient to complete the proof, because this implies the risk neutrality, which contradicts the fact that the variance of $F(\cdot)$ is subtracted in the definition of $U(\cdot)$.)

6.C.3 Suppose first that condition (i) holds. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Let $F(\cdot)$ be the distribution that puts probability 1/2 on $x - \varepsilon$ and on $x + \varepsilon$, and $F_\varepsilon(\cdot)$ be the distribution that puts probability $1/2 - \pi(x,\varepsilon,u)$ on $x - \varepsilon$ and $1/2 + \pi(x,\varepsilon,u)$ on $x + \varepsilon$. That is,

$$F(z) = \begin{cases} 0 & \text{if } z < x - \varepsilon, \\ 1/2 & \text{if } x - \varepsilon \le z < x + \varepsilon, \\ 1 & \text{if } x + \varepsilon \ge z. \end{cases}$$

$$F_\varepsilon(z) = \begin{cases} 0 & \text{if } z < x - \varepsilon, \\ 1/2 - \pi(x,\varepsilon,u) & \text{if } x - \varepsilon \le z < x + \varepsilon, \\ 1 & \text{if } x + \varepsilon \ge z. \end{cases}$$

Then $\int z dF(z) = x$ and $\int u(z)dF(z) \le u(x) = \int u(z)dF_\varepsilon(z)$ by (i). But

$$\int u(z)dF(z) = (1/2)u(x - \varepsilon) + (1/2)u(x + \varepsilon),$$

$$\int u(z)dF_\varepsilon(z) = (1/2 - \pi(x,\varepsilon,u))u(x - \varepsilon) + (1/2 + \pi(x,\varepsilon,u))u(x + \varepsilon).$$

$$= (1/2)u(x - \varepsilon) + (1/2)u(x + \varepsilon) + \pi(x,\varepsilon,u))(u(x + \varepsilon) - u(x - \varepsilon)).$$

Since $u(x + \varepsilon) - u(x - \varepsilon) > 0$, the above inequality is equivalent to $\pi(x,\varepsilon,u) \ge 0$. Thus (i) implies (iv).

Suppose conversely that condition (iv) holds. Let $y \in \mathbb{R}$, $z \in \mathbb{R}$, and $y > z$. Define $x = (y + z)/2$ and $\varepsilon = (y - z)/2$, then $y = x + \varepsilon$, $z = x - \varepsilon$, and

$$u(x) = (1/2 + \pi(x,\varepsilon,u))u(y) + (1/2 - \pi(x,\varepsilon,u))u(z)$$

$$= (1/2)u(y) + (1/2)u(z) + \pi(x,\varepsilon,u)(u(y) - u(z)).$$

Since $\pi(x,\varepsilon,u) \ge 0$ and $u(y) \ge u(z)$, this implies

$$(1/2)u(y) + (1/2)u(z) \le u(x) = u((1/2)y + (1/2)z).$$

Although we omit the proof, this is sufficient for the concavity of $u(\cdot)$.

Hence (iv) implies (ii). Since the equivalence of (i), (ii), and (iii) have already been established, this completes the proof of the equivalence of all four conditions.

6.C.4 **(a)** Let $\alpha = (\alpha_1,\ldots,\alpha_N) \in \mathbb{R}^N_+$, $\alpha' = (\alpha'_1,\ldots,\alpha'_N) \in \mathbb{R}^N_+$, and $\alpha \geq \alpha'$, then $\sum_n \alpha_n z_n \geq \sum_n \alpha'_n z_n$ for almost every realization $(z_1,\ldots,z_N)$, because all the returns are nonnegative with probability one. Since $u(\cdot)$ is increasing, this implies that $u(\sum_n \alpha_n z_n) \geq u(\sum_n \alpha'_n z_n)$ with probability one. Hence

$$\int u(\textstyle\sum_n \alpha_n z_n) dF(z_1,\ldots,z_N) \geq \int u(\sum_n \alpha'_n z_n) dF(z_1,\ldots,z_N),$$

that is, $U(\alpha) \geq U(\alpha')$.

(b) Let $\alpha = (\alpha_1,\ldots,\alpha_N) \in \mathbb{R}^N_+$, $\alpha' = (\alpha'_1,\ldots,\alpha'_N) \in \mathbb{R}^N_+$, and $\lambda \in [0,1]$, then, by the concavity of $u(\cdot)$,

$$\begin{aligned} u(\textstyle\sum_n (\lambda\alpha_n + (1-\lambda)\alpha'_n) z_n) &= u(\lambda \textstyle\sum_n \alpha_n z_n + (1-\lambda)\sum_n \alpha'_n z_n) \\ &\geq \lambda u(\textstyle\sum_n \alpha_n z_n) + (1-\lambda) u(\sum_n \alpha'_n z_n) \end{aligned}$$

for almost every realization $(z_1,\ldots,z_n)$. Hence

$$\begin{aligned} &U(\lambda\alpha + (1-\lambda)\alpha') \\ &= \int u(\textstyle\sum_n (\lambda\alpha_n + (1-\lambda)\alpha'_n) z_n) dF(z_1,\ldots,z_N) \\ &\geq \int (\lambda u(\textstyle\sum_n \alpha_n z_n) + (1-\lambda) u(\sum_n \alpha'_n z_n)) dF(z_1,\ldots,z_N) \\ &= \lambda \int u(\textstyle\sum_n \alpha_n z_n) dF(z_1,\ldots,z_N) + (1-\lambda)\int u(\sum_n \alpha'_n z_n)) dF(z_1,\ldots,z_N) \\ &= \lambda U(\alpha) + (1-\lambda) U(\alpha'). \end{aligned}$$

(c) Let $(\alpha^m)_{m\in\mathbb{N}}$ be a sequence in $\mathbb{R}^N_+$ converging to $\alpha \in \mathbb{R}^N_+$, then there exists a positive number B such that $\alpha^m \leq (B,\ldots,B)$ for every m. Of course, $U(B,\ldots,B)$ is finite. But this is equivalent to saying that the (measurable) function $z \to u(\sum_n B z_n)$ is integrable. Since $u(\cdot)$ is monotone and all the returns are nonnegative with probability one, $u(\sum_n \alpha^m_n z_n) \leq u(\sum_n B z_n)$ for every m and for every realization $(z_1,\ldots,z_N)$. Moreover, since $u(\cdot)$ is continuous, $u(\sum_n \alpha^m_n z_n)$

converges to $u(\sum_n \alpha_n z_n)$ for almost every realization $(z_1,\ldots,z_N)$. Hence, by Lebesgue's dominated convergence theorem,

$$\int u(\sum_n \alpha_n^m x_n) dF(x_1,\ldots,x_N) \to \int u(\sum_n \alpha_n x_n) dF(x_1,\ldots,x_N).$$

That is, $U(\alpha^m) \to U(\alpha)$.

6.C.5 **(a)** Let $x \in \mathbb{R}^L_+$, $y \in \mathbb{R}^L_+$ and $\lambda \in [0,1]$. In analogy with expression (6.C.1) the value $\lambda u(x) + (1 - \lambda)u(y)$ can be considered as the expected utility from the lottery that yields x with probability λ and y with probability $1 - \lambda$. On the other hand, the value $u(\lambda x + (1 - \lambda)y)$ is the utility from consuming the mean $\lambda x + (1 - \lambda)y$ of the lottery with probability one. The concavity of $u(\cdot)$ would then imply that consuming the mean bundle of the L commodities with probability one is at least as good as entering into the lottery. But this is the defining property of risk aversion in Definition 6.C.1.

(b) [First printing errata: The Bernoulli utility function $u(\cdot)$ for wealth should be denoted by another symbol, say $\tilde{u}(\cdot)$, to avoid confusion with the original utility function $u(\cdot)$ defined on $\mathbb{R}^L_+$.] Let $p \gg 0$ be a fixed price vector, w and w' be two wealth levels, and $\lambda \in [0,1]$. Denote the demand function by $x(\cdot)$ and let $x = x(p,w)$ and $x' = x(p,w')$, then $p\cdot(\lambda x + (1 - \lambda)x') \le \lambda w + (1 - \lambda)w'$. Thus $u(\lambda x + (1 - \lambda)x') \le \tilde{u}(\lambda w + (1 - \lambda)w')$. If $u(\cdot)$ is concave, then

$$u(\lambda x + (1 - \lambda)x') \ge \lambda u(x) + (1 - \lambda)u(x') = \lambda\tilde{u}(w) + (1 - \lambda)\tilde{u}(w').$$

Hence $\tilde{u}(\lambda w + (1 - \lambda)w') \ge \lambda\tilde{u}(w) + (1 - \lambda)\tilde{u}(w')$. Thus $\tilde{u}(\cdot)$ also exhibits risk aversion.

The following interpretation can be given to this result. Although, in the text, we are mainly concerned with the cases where outcomes are monetary amounts, in many cases in economic theory, utilities do not directly come from money, but from physical commodities. It is therefore desirable to *derive* risk aversion of Bernoulli utility functions for money from the properties of the underlying utility function for the commodities. The above result says that, if an individual has a risk-averse utility function for commodities, the his Bernoulli utility functions exhibits risk aversion.

(c) We shall give an example with the properties stated in the exercise. Let $L = 2$. Define $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ by $u(x) = \sqrt{\text{Max}\{x_1, x_2\}}$, then $u(\cdot)$ is not concave. Now consider the price vector $p = (1,2)$, then, for each $w \geq 0$, $x(p,w) = (w,0)$. Hence $\tilde{u}(w) = \sqrt{w}$, which is concave and exhibits risk aversion. The lesson from this example is that, in order to obtain the risk aversion of $\tilde{u}(\cdot)$ for a fixed price vector, all that matters is the risk attitude along the wealth expansion path.

6.C.6 **(a)** Suppose that condition (ii) is true and let $F(\cdot)$ be any distribution function, then

$$\psi(u_1(c(F,u_2))) = u_2(c(F,u_2)) = \int u_2(x)dF(x).$$

Since $\psi(\cdot)$ is concave,

$$\int u_2(x)dF(x) = \int \psi(u_1(x))dF(x) \leq \psi(\int u_1(x)dF(x)).$$

Thus $\psi(u_1(c(F,u_2))) \leq \psi(\int u_1(x)dF(x))$. Since $\psi(\cdot)$ is increasing, this implies that $u_1(c(F,u_2)) \leq \int u_1(x)dF(x)$. Since $\int u_1(x)dF(x) = u_1(c(F,u_1))$, this implies that $u_1(c(F,u_2)) \leq u_1(c(F,u_1))$. Since $u_1(\cdot)$ is increasing, we obtain $c(F,u_2) \leq c(F,u_1)$. Condition (iii) is thus established.

Conversely, suppose that (iii) is true. Let $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $\lambda \in [0,1]$. We shall prove that

$$\psi(\lambda u_1(x) + (1 - \lambda)u_1(y)) \geq \lambda\psi(u_1(x)) + (1 - \lambda)\psi(u_1(y)).$$

Let $F(\cdot)$ be the distribution function that puts probability λ on x and probability $1 - \lambda$ on y. Then, $\lambda u_1(x) + (1 - \lambda)u_1(y) = u_1(c(F,u_1))$ and hence $\psi(\lambda u_1(x) + (1 - \lambda)u_1(y)) = u_2(c(F,u_1))$. On the other hand, by the definition,

$$\lambda\psi(u_1(x)) + (1 - \lambda)\psi(u_1(y)) = \lambda u_2(x) + (1 - \lambda)u_2(y) = u_2(c(F,u_2)).$$

By (iii) and the increasingness of $u_2(\cdot)$, we obtain

$$\psi(\lambda u_1(x) + (1 - \lambda)u_1(y)) \geq \lambda\psi(u_1(x)) + (1 - \lambda)\psi(u_1(y)).$$

(b) Suppose first that condition (iii) holds. If $\int u_2(x)dF(x) \geq u_2(\bar{x})$, then $u_2(c(F,u_2)) \geq u_2(\bar{x})$. Thus $c(F,u_2) \geq \bar{x}$. By condition (iii), $c(F,u_1) \geq \bar{x}$. Hence $u_1(c(F,u_1)) \geq u_1(\bar{x})$, or $\int u_1(x)dF(x) \geq u_1(\bar{x})$. Thus condition (v) holds.

Suppose next that (v) holds, then $\int u_1(x)dF(x) \geq u_1(c(F,u_2))$. Since $\int u_1(x)dF(x) = u_1(c(F,u_1))$, we have $u_1(c(F,u_1)) \geq u_1(c(F,u_2))$ and hence $c(F,u_1) \geq c(F,u_2)$.

6.C.7 Suppose first that condition (iii) holds. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Denote by $F(\cdot)$ the distribution function that puts probability $1/2 - \pi(x,\varepsilon,u_2)$ on $x - \varepsilon$ and $1/2 + \pi(x,\varepsilon,u_2)$ on $x + \varepsilon$. That is,

$$F(z) = \begin{cases} 0 & \text{if } z < x - \varepsilon, \\ 1/2 - \pi(x,\varepsilon,u_2) & \text{if } x - \varepsilon \leq z < x + \varepsilon, \\ 1 & \text{if } x + \varepsilon \geq z. \end{cases}$$

Then $c(F,u_2) = x$. By (iii), $c(F,u_1) \geq x$. Thus $u_1(c(F,u_1)) \geq u_1(x)$. But here, we have

$$\begin{aligned} &u_1(c(F,u_1)) \\ &= (1/2 - \pi(x,\varepsilon,u_2))u_1(x - \varepsilon) + (1/2 + \pi(x,\varepsilon,u_2))u_1(x + \varepsilon) \\ &= (1/2)u_1(x - \varepsilon) + (1/2)u_1(x + \varepsilon) + \pi(x,\varepsilon,u_2)(u_1(x + \varepsilon) - u_1(x - \varepsilon)) \end{aligned}$$

and

$$\begin{aligned} &u_1(x) \\ &= (1/2 - \pi(x,\varepsilon,u_1))u_1(x - \varepsilon) + (1/2 + \pi(x,\varepsilon,u_1))u_1(x + \varepsilon) \end{aligned}$$

$$= (1/2)u_1(x - \varepsilon) + (1/2)u_1(x + \varepsilon) + \pi(x,\varepsilon,u_1)(u_1(x + \varepsilon) - u_1(x - \varepsilon)).$$

Thus the last inequality is equivalent to $\pi(x,\varepsilon,u_2) \geq \pi(x,\varepsilon,u_1)$. Hence condition (iv) holds.

Suppose now that condition (iv) holds. Since $\pi(x,0,u_1) = \pi(x,0,u_2) = 0$, (iv) implies that $\partial\pi(x,0,u_2)/\partial\varepsilon \geq \partial\pi(x,0,u_2)/\partial\varepsilon$. Since $r_A(x,u_1) = 4\partial\pi(x,0,u_1)/\partial\varepsilon$ and $r_A(x,u_2) = 4\partial\pi(x,0,u_2)/\partial\varepsilon$, (i) follows.

6.C.8 Let w_1 and w_2 be two wealth levels such that $w_1 > w_2$ and define $u_1(z) = u(w_1 + z)$ and $u_2(z) = u(w_2 + z)$, then $u_2(\cdot)$ is a concave transformation of $u_1(\cdot)$ by Proposition 6.C.3. It was shown in Example 6.C.2 continued that the demand for the risky asset of $u_1(\cdot)$ is greater than that of $u_2(\cdot)$. This means that the demand for the risky asset of $u(\cdot)$ is greater at wealth level w_1 than at w_2.

6.C.9 [First printing errata: The function $u(\cdot)$ on the left-hand side of the equality on the fifth line should be denoted by a different symbol, because, on the right-hand side, $u(\cdot)$ is used for the utility function on the first period.]

(a) The first-order condition for the first problem is $u'(w - x_0) = v'(x_0)$. For the second problem, let's first define a function $\phi(\cdot)$ by

$$\phi(x) = u(w - x) + E[v(x + y)].$$

Then $\phi'(x) = - u'(w - x) + E[v'(x + y)]$ and $\phi''(x) = u''(w - x) + E[v''(x + y)]$. Note also that $\phi'(x^*) = 0$ and $\phi''(x) \leq 0$ for every x, which implies that if $\phi'(x) > 0$, then $x^* > x$. Now, since $E[v'(x_0 + y)] > v'(x_0)$,

$$\phi'(x_0) = - u'(w - x_0) + E[v'(x_0 + y)] = - v'(x_0) + E[v'(x_0 + y)] > 0.$$

Hence $x^* > x_0$.

(b) Define two functions $\eta_1(\cdot)$ and $\eta_2(\cdot)$ by $\eta_1(x) = - v_1'(x)$ and $\eta_2(x) = - v_2'(x)$. Then $\eta_1(\cdot)$ and $\eta_2(\cdot)$ are increasing and the coefficients of absolute prudence of $v_1(\cdot)$ and of $v_2(\cdot)$ are equal to the coefficients of absolute risk aversion of $\eta_1(\cdot)$ and of $\eta_2(\cdot)$. Thus, if the coefficient of absolute prudence of $v_1(\cdot)$ is not larger than that of $v_2(\cdot)$, then the coefficient of absolute risk aversion of $\eta_1(\cdot)$ is not larger than that of $\eta_2(\cdot)$. Moreover, since $E[v_1'(x_0 + y)] > v_1'(x_0)$, we have $E[\eta_1(x_0 + y)] < \eta_1(x_0)$. Thus, by applying Proposition 6.C.2 to $\eta_1(\cdot)$ and $\eta_2(\cdot)$, we obtain $E[\eta_2(x_0 + y)] < \eta_2(x_0)$. Hence $E[v_2'(z_0 + y)] > v_2'(z_0)$.

The implication of this fact to part (a) is that, if the coefficient of absolute prudence of the first is not larger than that of the second, and if the risk y induces the first individual to save more, then it also induces the second to do so. Hence coefficients of absolute prudence measure how much individuals are willing to save when faced with a risk in the future.

(c) If $v'''(x) > 0$, then $\eta''(x) = - v'(x) < 0$ and hence $\eta(\cdot)$ exhibits risk aversion. Thus $E[\eta(x + y)] < \eta(x)$, that is, $E[v_1'(x + y)] > v_1'(x)$.

(d) Since

$$r_A'(x,v) = - \frac{v'''(x)v'(x) - v''(x)^2}{v'(x)^2} = \frac{v''(x)}{v'(x)}\left(- \frac{v'''(x)}{v''(x)} + \frac{v''(x)}{v'(x)}\right) < 0,$$

the assertion follows.

6.C.10 Throughout this answer, we let x_1 and x_2 be two fixed wealth levels such that $x_1 > x_2$ and define $u_1(z) = u(x_1 + z)$ and $u_2(z) = u(x_2 + z)$. It is sufficient to prove that each of the five conditions of Proposition 6.C.3 is equivalent to its counterpart of Proposition 6.C.2.

Since $r_A(z,u_1) = r_A(x_1 + z, u)$ and $r_A(z,u_2) = r_A(x_2 + z, u)$, property (i) of Proposition 6.C.3 is equivalent to (i) of Proposition 6.C.2.

Property (ii) of Proposition 6.C.3 is nothing but a restatement of (ii) of Proposition 6.C.2.

As for property (iii), note that

$$\int u_1(z)dF(z) = \int u(x_1 + z)dF(z) = u(c_{x_1}) = u((c_{x_1} - x_1) + x_1) = u_1(c_{x_1} - x_1)$$

and likewise for $u_2(\cdot)$. Thus the certainty equivalent for $u_1(\cdot)$ is smaller than that for $u_2(\cdot)$ if and only if $c_{x_1} - x_1 < c_{x_2} - x_2$. Thus property (iii) of Proposition 6.C.3 is equivalent to (iii) of Proposition 6.C.2.

As for property (iv), since

$$u(x_1) = (1/2 - \pi(x_1,\varepsilon,u))u(x_1 - \varepsilon) + (1/2 + \pi(x_1,\varepsilon,u))u(x_1 + \varepsilon),$$

we have

$$u_1(0) = (1/2 - \pi(x_1,\varepsilon,u))u_1(-\varepsilon) + (1/2 + \pi(x_1,\varepsilon,u))u_1(\varepsilon).$$

Hence $\pi(x_1,\varepsilon,u) = \pi(0,\varepsilon,u_1)$. Similarly, $\pi(x_2,\varepsilon,u) = \pi(0,\varepsilon,u_2)$. Hence (iv) of Proposition 6.C.3 is equivalent to (iv) of Proposition 6.C.2.

Note that $\int u(x_1 + z)dF(z) \geq u(x_1)$ if and only if $\int u_1(z)dF(z) \geq u_1(0)$, and likewise for $u_2(\cdot)$. Thus property (v) of Proposition 6.C.3 is equivalent to (v) of Proposition 6.C.2.

6.C.11 For any wealth level x, denote by $\gamma(x)$ the optimal proportion of x invested in the risky asset. We shall give a direct proof that if the coefficient of relative risk aversion is increasing, then $\gamma'(x) < 0$; along the same line of proof, we can show that if it is decreasing, then $\gamma'(x) > 0$. As shown in Exercise 6.C.2, $\gamma(x)$ is positive and satisfies the following first-order condition for every x:

$$\int u'((1 - \gamma(x) + \gamma(x)z)x)(z - 1)x dF(z) = 0.$$

Hence

$$\gamma'(x) = \frac{-\int u''((1-\gamma(x)+\gamma(x)z)x)(1-\gamma(x)+\gamma(x)z)(z-1)x\,dF(z)}{\int u''((1-\gamma(x)+\gamma(x)z)x)(z-1)^2x^2\,dF(z)}.$$

Since the denominator is negative, it is sufficient to show that the numerator is positive.

By the definition of the coefficient of relative risk aversion,

$$- u''((1-\gamma(x)+\gamma(x)z)x)(1-\gamma(x)+\gamma(x)z)x$$

$$= r_R((1-\gamma(x)+\gamma(x)z)x)u'((1-\gamma(x)+\gamma(x)z)x)$$

for every realization z. Note also that if $z > 1$, then $(1-\gamma(x)+\gamma(x)z)x > x$ by $\gamma(x) > 0$. Since the coefficient of relative risk aversion is increasing, this implies that $r_R((1-\gamma(x)+\gamma(x)z)x) > r_R(x)$. Hence

$$- u''((1-\gamma(x)+\gamma(x)z)x)(1-\gamma(x)+\gamma(x)z)x$$

$$> r_R(x)u'((1-\gamma(x)+\gamma(x)z)x).$$

By $z - 1 > 0$,

$$- u''((1-\gamma(x)+\gamma(x)z)x)(1-\gamma(x)+\gamma(x)z)x(z-1)$$

$$> r_R(x)u'((1-\gamma(x)+\gamma(x)z)x)(z-1).$$

We can similarly show that this last inequality also holds for every $z < 1$. Therefore,

$$- \int u''((1-\gamma(x)+\gamma(x)z)x)(1-\gamma(x)+\gamma(x)z)x(z-1)dF(z)$$

$$> \int r_R(x)u'((1-\gamma(x)+\gamma(x)z)x)(z-1)dF(z)$$

$$= r_R(x)\int u'((1-\gamma(x)+\gamma(x)z)x)(z-1)dF(z) = 0$$

by the first-order condition.

6.C.12 **(a)** [First printing errata: The coefficient β should be positive if $\rho < 1$ and negative if $\rho > 1$. This makes $u(\cdot)$ increasing.] It is easy to check that, if $u(x) = \beta x^{1-\rho} + \gamma$ with $\rho \neq 1$ and $\gamma \in \mathbb{R}$, then $u(\cdot)$ exhibits constant relative risk aversion ρ. Suppose conversely that $u(\cdot)$ exhibits constant risk aversion ρ, then $u''(x)/u'(x) = -\rho/x$. Thus $\ln(u'(x)) = -\rho \ln x + c_1$ for some

$c_1 \in \mathbb{R}$. Thus $u'(x) = (\exp c_1)x^{-\rho}$. Hence $u(x) = (\exp c_1)x^{1-\rho}/(1 - \rho) + c_2$ for some $c_2 \in \mathbb{R}$. Letting $\beta = (\exp c_1)/(1 - \rho)$ and $\gamma = c_2$, we complete the proof.

(b) It is easy to check that, if $u(x) = \beta \ln x + \gamma$ with $\beta > 0$ and $\gamma \in \mathbb{R}$, then $u(\cdot)$ exhibits constant relative risk aversion one. The other direction can be shown in the same way as in (a).

(c) By L'hopital's rule,

$$\lim_{\rho \to 1} [(x^{1-\rho} - 1)/(1 - \rho)] = \lim_{\rho \to 1} (-\ln x)x^{1-\rho}/(-1) = \ln x.$$

6.C.13 Let $\pi(\cdot)$ be the profit function and $F(\cdot)$ be the distribution function of the random price. Since $\pi(\cdot)$ is convex, $\int \pi(p)dF(p) \geq \pi(\int p dF(p))$ by Jensen's inequality. But the left-hand side is the expected payoff from the uncertain prices and the right-hand side is the utility of the expected price vector. Thus the firm prefers the uncertain prices.

6.C.14 Define a function $g(\cdot)$ by $g(\alpha) = k\alpha + v(u^{-1}(\alpha))$, then $g(u(x)) = ku(x) + v(x) = u^*(x)$. It is thus sufficient to show that $g(\cdot)$ is concave. For this, in turn, it is sufficient to prove that $(v \circ u^{-1})(\cdot)$ is concave.

Let $\alpha, \beta \in \mathbb{R}$ and $\lambda \in [0,1]$. Since $u(\cdot)$ is increasing and concave, $u^{-1}(\cdot)$ is convex. Thus

$$u^{-1}(\lambda\alpha + (1 - \lambda)\beta) \leq \lambda u^{-1}(\alpha) + (1 - \lambda)u^{-1}(\beta).$$

Since $v(\cdot)$ is nonincreasing, this implies

$$v(u^{-1}(\lambda\alpha + (1 - \lambda)\beta)) \geq v(\lambda u^{-1}(\alpha) + (1 - \lambda)u^{-1}(\beta)).$$

Since $v(\cdot)$ is concave,

$$v(\lambda u^{-1}(\alpha) + (1 - \lambda)u^{-1}(\beta)) \geq \lambda v(u^{-1}(\alpha)) + (1 - \lambda)v(u^{-1}(\beta)).$$

Thus,

$$v(u^{-1}(\lambda\alpha + (1 - \lambda)\beta)) \geq \lambda v(u^{-1}(\alpha)) + (1 - \lambda)v(u^{-1}(\beta)).$$

or, equivalently,

$$(v \circ u^{-1})(\lambda\alpha + (1 - \lambda)\beta) \geq \lambda(v \circ u^{-1})(\alpha) + (1 - \lambda)(v \circ u^{-1})(\beta).$$

(b) [First printing errata: The entire interval $[0, +\infty]$ should be $[0, +\infty)$.] Suppose that we have $u^*(x) = ku(x) + v(x)$ for a non-constant $v(\cdot)$. Since $v(\cdot)$ is decreasing and concave, $v(x + 1) - v(x)$ is negative and decreasing with x. On the other hand, since $u(\cdot)$ is increasing, concave, and bounded above, $u(x + 1) - u(x)$ is positive and decreasing, and converges to zero. Since

$$u^*(x + 1) - u^*(x) = k(u(x + 1) - u(x)) + (v(x + 1) - v(x)),$$

$u^*(x + 1) - u^*(x)$ is negative for any sufficiently large x. That is, $u^*(\cdot)$ is not increasing around such x. But this is a contradiction to the assumption that $u^*(\cdot)$ is increasing. Thus, if $u(\cdot)$ is bounded, then there is no non-constant $v(\cdot)$ such that $u^*(x) = ku(x) + v(x)$ for all $x \in [0, +\infty)$.

(c) By (a) and (b), it is sufficient to find $u(\cdot)$ and $u^*(\cdot)$ such that $u^*(\cdot)$ is more risk averse (in the Arrow-Pratt sense) than $u(\cdot)$ and $u(\cdot)$ is bounded. Define $u(x) = -\exp(-\alpha x)$ and $u^*(x) = -\exp(-\beta x)$, where $0 < \alpha < \beta$. By Example 6.C.4, $u(\cdot)$ and $u^*(\cdot)$ exhibit constant absolute risk aversion with coefficients α and β. Hence, $u^*(\cdot)$ is more risk averse than $u(\cdot)$, but, since $u(x) < 0$ for all x, $u^*(\cdot)$ is not strongly more risk averse than $u(\cdot)$.

6.C.15 Throughout this answer, we assume that $a \neq b$, because, otherwise, there would be no uncertainty involved in the payment of the second asset.

(a) If $\text{Min}\{a,b\} \geq 1$, the risky asset pays at least as high a return as the riskless asset at both states, and a strictly higher return at one of them. Then all the wealth is invested to the risky asset. Thus, $\text{Min}\{a,b\} < 1$ is a necessary condition for the demand for the riskless asset to be strictly positive.

(b) If $\pi a + (1 - \pi)b \leq 1$, then the expected return does not exceed the payments of the riskless asset and hence the risk-averse decision maker does not demand the risky asset at all. Thus, $\pi a + (1 - \pi)b > 1$ is a necessary condition for the demand for the risky asset to be strictly positive.

In the following answers, we assume that the demands for both assets are always positive.

(c) Since the prices of the two assets are equal to one, their marginal utilities must be equal. Thus

$$\pi u'(x_1 + x_2 a) + (1 - \pi)u'(x_1 + x_2 b) = \pi a u'(x_1 + x_2 a) + (1 - \pi)b u'(x_1 + x_2 b).$$

That is,

$$\pi(1 - a)u'(x_1 + x_2 a) + (1 - \pi)(1 - b)u'(x_1 + x_2 b) = 0.$$

This and $x_1 + x_2 = 1$ constitute the first-order condition.

(d) Taking b as constant, define

$$\phi(a,\pi,x_1) = \pi(1 - a)u'(x_1 + (1 - x_1)a) + (1 - \pi)(1 - b)u'(x_1 + (1 - x_1)b),$$

then

$$\partial\phi/\partial a = -\pi u'(x_1 + (1 - x_1)a) + \pi(1 - a)(1 - x_1)u''(x_1 + (1 - x_1)a) < 0,$$

$$\partial\phi/\partial x_1 = \pi(1 - a)^2 u''(x_1 + (1 - x_1)a) + (1 - \pi)(1 - b)^2 u''(x_1 + (1 - x_1)b) < 0.$$

Thus, by the implicit function theorem (Theorem M.E.1),

$$dx_1/da = -\frac{\partial\phi/\partial a}{\partial\phi/\partial x_1} < 0.$$

(e) It follows from the condition of (b) that b > 1, that is, that a is the worse outcome of the risky asset. Thus, if the probability π of the worse outcome is increased, then it is anticipated that the demand for the riskless asset is increased.

(f) Since $b > 1$,

$$\partial\phi/\partial\pi = (1 - a)u'(x + (1 - x)a) - (1 - b)u'(x + (1 - x)b)$$

$$= (1 - a)u'(x + (1 - x)a) + (b - 1)u'(x + (1 - x)b) > 0,$$

because $a < 1 < b$. Thus $dx/d\pi = -\dfrac{\partial\phi/\partial\pi}{\partial\phi/\partial x} > 0$, as anticipated.

6.C.16 Throughout the answer, we assume that $u(\cdot)$ is continuous, so that the maximum and the minimum are attained.

(a) If the individual owns the lottery, his random wealth is $(w + G, w + B)$. Thus the minimal selling price R_s is defined by

$$pu(w + G) + (1 - p)u(w + B) = u(w + R_s).$$

(b) If he buys the lottery at price R, his random wealth is $(w - R + G, w - R + B)$. The maximal buying price R_b is defined by

$$pu(w - R_b + G) + (1 - p)u(w - R_b + B) = u(w).$$

(c) In general, these two prices are different. However, if $u(\cdot)$ exhibits constant absolute risk aversion, then they are the same. In fact, the above two equations can be restated as $c_w = w + R_s$ and $c_{w-R_b} = w$, where c_w and c_{w-R_b} are defined as in (iii) of Proposition 6.C.3. According to the proposition, the constant absolute risk aversion implies that

$$w - c_w = (w - R_b) - c_{w-R_b}.$$

This is equivalent to $R_s = R_b$.

(d) By a direct calculation,

$$R_s = 5[(7 - 4\sqrt{3})p^2 + (4\sqrt{3} - 6)p + 1],$$

and R_b is one of the solutions to the quadratic equation

$$(1 - 2p^2)R_b^2 - 10(2p^3 + 7p^2 - 8p + 1)R_b - 25(23p^2 - 54p + 29) = 0.$$

6.C.17 According to Exercise 6.C.12, if $u(\cdot)$ exhibits constant relative risk aversion ρ, then $u(x) = \beta x^{1-\rho} + \gamma$ or $u(x) = \beta \ln(x) + \gamma$. In this answer, we assume $u(x) = \beta x^{1-\rho}$. The case of $\beta \ln(x) + \gamma$ can be proven by the same argument. Let's first consider the portfolio problem of the individual in period t = 1, after a realization of the random return has generated wealth level w_1. Denoting the distribution function of the return by $F(\cdot)$, his problem is

$$\text{Max}_{0 \le \alpha_1 \le 1} \int u(((1 - \alpha_1)R + \alpha_1 x_2)w_1)dF(x_2).$$

As discussed in Example 6.C.2 continued (and also in Exercise 6.C.11), we can show that the solution does not depend on the value of w_1. Denote the solution by α^*. If he chooses portfolio α_0 at t = 0, then his random wealth at t = 1 is $w_1 = ((1 - \alpha_0)R + \alpha_0 x_1)w_0$. Given the solution α^* at t = 1, his problem in period t = 0 is

$$\text{Max}_{0 \le \alpha_1 \le 1} \int\int u(((1 - \alpha^*)R + \alpha^* x_2)((1 - \alpha_0)R + \alpha_0 x_1)w_0)dF(x_2)dF(x_1).$$

Since the distributions of x_1 and x_2 are independent and $u(x) = \beta x^{1-\rho}$, we can rewrite the objective function as

$$[\int((1 - \alpha^*)R + \alpha^* x_2)^{1-\rho}dF(x_2)][\int u(((1 - \alpha_0)R + \alpha_0 x_1)w_0)dF(x_1)].$$

Since the first integral does not depend on the choice of α_0, the solution is again $\alpha_0 = \alpha^*$. This completes the proof.

For the case of a utility function exhibits constant absolute risk aversions, the absolute amounts of wealth invested on the risky asset may vary over the two periods t = 0,1, but those in period t = 1 do not depend on the realization of x_1. To see this, let $u(x) = -\beta e^{-\rho}$. The individual's problem at t = 1 is

$$\text{Max}_{0\le\alpha_1\le w} \int u((w_1 - \alpha_1)R + \alpha_1 x_2)dF(x_2).$$

The solution turns out to be independent of the value of w_1, and hence of x_0. Denote the solution by α^*. If he chooses portfolio α_0 at t = 0, then his random wealth at t = 1 is $w_1 = (w_0 - \alpha_0)R + \alpha_0 x_1$. Given the solution α^* at t = 1, his problem in period t = 0 is

$$\text{Max}_{0\le\alpha_1\le 1} \int\int u(((w_0 - \alpha_0)R + \alpha_0 x_1 - \alpha^*)R + \alpha^* x_2)dF(x_2)dF(x_1).$$

Since the distributions of x_1 and x_2 are independent and $u(x) = -\beta e^{-\rho}$, we can rewrite the objective function as

$$[\int \exp(-\alpha^* R + \alpha^* x_2)dF(x_2)][\int -\beta\exp(-((w_0 - \alpha_0)R + \alpha_0 x_1)R\rho)dF(x_1)].$$

Since the first integral does not depend on the choice of α_0, the solution of this maximization problem is the same as the solution of the problem of maximizing

$$\int -\beta\exp(-((w_0 - \alpha_0)R + \alpha_0 x_1)R\rho)dF(x_1).$$

But the latter is the same as what the consumer would choose at t = 1 if his coefficient of absolute risk aversion is equal to $R\rho$.

Now, if R = 1, then the consumer invests a constant absolute amount of wealth over two periods. Thus, their proportions out of the total wealths are larger if the total wealths are smaller. So, the proportion α_1/w_1 now depends on w_1 and hence on the realization x_1. Hence the proportions can no longer be constant.

6.C.18 **(a)** A direct calculation shows that the coefficient of absolute risk aversion at w = 5 is 0.1. Exercise 6.C.12(a) shows that the coefficient of relative risk aversion is 0.5, which is constant over w.

(b) By a direct calculation, the certainty equivalent is 9 and the probability premium is $(\sqrt{10} - 3)/2$.

(c) By a direct calculation, the certainty equivalent is 25 and the probability premium is $(\sqrt{26} - 5)/2$.

For each of these two lotteries, the difference between the mean of the lottery and the certainty equivalent is equal to one. However, the probability premium for the first lottery is larger. This is because $u(\cdot)$ exhibits constant relative risk aversion and hence decreasing absolute risk aversion.

6.C.19 Foe each n, denote by β_n the wealth invested in risky asset n. The wealth invested in the riskless asset is then $w - \sum_n \beta_n$. If the individual takes portfolio $\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{R}^N$, then his random consumption is $x = (w - \sum_n \beta_n)r + \sum_n \beta_n z_n$, where z_n denotes the random return of asset n. By linearity of normal distributions, x is a normal distribution with mean $(w - \sum_n \beta_n)r + \sum_n \beta_n \mu_n$ and variance $\beta \cdot V\beta$. The expected utility from x is $E[- \exp(- \alpha x)]$. But this is equal to the value, multiplied by - 1, at $- \alpha$ of the moment-generating function of the normal distribution with mean $(w - \sum_n \beta_n)r + \sum_n \beta_n \mu_n$ and variance $\beta \cdot V\beta$. Therefore,

$$E[- \exp(- \alpha x)] = - \exp[((w - \textstyle\sum_n \beta_n)r + \sum_n \beta_n \mu_n)(- \alpha) - (\beta \cdot V\beta)(- \alpha)^2/2].$$

By applying the monotone transformation $u \to (- 1/\alpha)\ln(- u)$ to this utility function, we obtain

$$((w - \textstyle\sum_n \beta_n)r + \sum_n \beta_n \mu_n) + (\beta \cdot V\beta)\alpha/2.$$

The first-order condition for a maximum of this objective function with respect to β gives the optimal portfolio $\beta^* = \alpha^{-1}V^{-1}(\mu - re)$, where e is the vector of $\mathbb{R}^N$ whose components are all equal to one.

6.C.20 For each $\varepsilon \geq 0$, let $F_\varepsilon(\cdot)$ be the distribution function of the lottery that pays x + ε with probability 1/2 and x - ε with probability 1/2. Then, $c(F_\varepsilon, u)$ is defined as the solution to the equation

$$(1/2)u(x + \varepsilon) + (1/2)u(x - \varepsilon) - u(c) = 0$$

with respect to c. Hence, by the implicit function theorem (Theorem M.E.1), $c(F_\varepsilon, u)$ is a differentiable function of ε and

$$(1/2)u'(x + \varepsilon) - (1/2)u'(x - \varepsilon) - u'(c(F_\varepsilon, u))(\partial c(F_\varepsilon, u)/\partial\varepsilon) = 0.$$

By putting ε = 0, we obtain $\partial c(F_0, u)/\partial\varepsilon = 0$. Also, by further differentiating the left-hand side of this equality with respect to ε, we obtain

$$(1/2)u''(x + \varepsilon) + (1/2)u''(x - \varepsilon)$$
$$- u''(c(F_\varepsilon, u))(\partial c(F_\varepsilon, u)/\partial\varepsilon)^2 - u'(c(F_\varepsilon, u))(\partial^2 c(F_\varepsilon, u)/\partial\varepsilon^2) = 0.$$

Thus, by putting ε = 0 and substituting $\partial c(F_0, u)/\partial\varepsilon = 0$, we obtain

$$u''(x) - u'(c(F_\varepsilon, u))(\partial^2 c(F_\varepsilon, u)/\partial\varepsilon^2) = 0.$$

Thus $\partial^2 c(F_\varepsilon, u)/\partial\varepsilon^2 = - r_A(x)$.

6.D.1 Let $L = (p_1, p_2, p_3)$ and $L' = (p'_1, p'_2 . p'_3)$ be two lotteries and $F(\cdot)$ and $G(\cdot)$ be their distribution functions.

(a) If a Bernoulli utility function is increasing, then there exists p ∈ (0,1) such that the decision maker is indifferent between the sure outcome of \$2 and the lottery that pays \$1 with probability p and \$3 with probability 1 - p. Thus, the indifference line that goes through the \$2-vertex must hit some point on the (\$1,\$3)-face (excluding the vertices) and all indifference lines must be parallel to it. Conversely, this condition implies that the Bernoulli utility function increasing. By varying p vary from 0 to 1, we can identify the area of the lotteries that are above all indifference curves going through L. The area is shaded in the following figure:

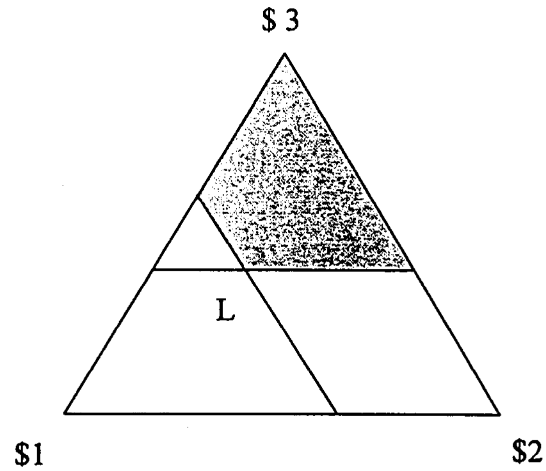


Figure 6.D.1(a)

Thus, $G(\cdot)$ first-order stochastically dominates $F(\cdot)$ if and only if L' is located above the segment that goes through L and is parallel to the (\$1,\$2)-face and also above the segment that goes through L and parallel to the (\$2,\$3)-face.

(b) The distribution $G(\cdot)$ first-order stochastically dominates $F(\cdot)$ if and only if $p_1 \geq p'_1$ and $p_1 + p_2 \geq p'_1 + p'_2$. Since the second inequality is equivalent to $p_3 \leq p'_3$, $G(\cdot)$ first-order stochastically dominates $F(\cdot)$ if and only if L' is located in the shaded area in the figure below:

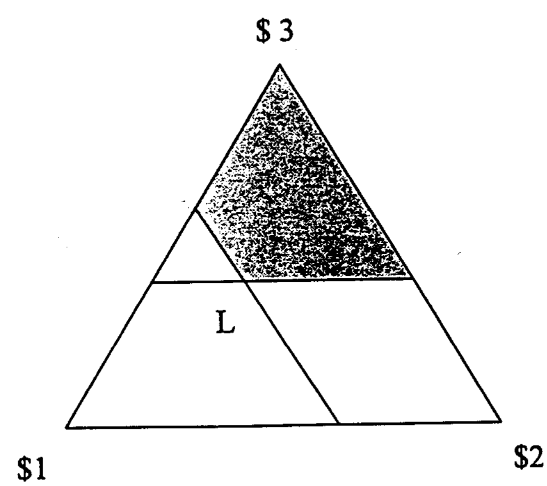


Figure 6.D.1(b)

6.D.2 [<u>First printing errata</u>: The phrase "the mean of x under $F(\cdot)$, $\int x dF(x)$, exceeds that under $G(\cdot)$, $\int x dG(x)$" should be "the mean of x under $G(\cdot)$, $\int x dG(x)$, cannot exceed that under $F(\cdot)$, $\int x dF(x)$". That is, the equality of the two means should be allowed.] For the first assertion, simply put $u(x) = x$ and apply Definition 6.D.1. As for the second, let $p \in (0,1/2)$ and consider the following two distributions:

$$F(z) = \begin{cases} 0 & \text{if } \quad z < 0, \\ p & \text{if } 0 \le z < 2, \\ 1 & \text{if } 2 \le z, \end{cases}$$

$$G(z) = \begin{cases} 0 & \text{if } z < 1, \\ 1 & \text{if } 1 \le z. \end{cases}$$

Then $F(1/2) = p > 0 = G(1/2)$ and $\int x dF(x) = 2(1 - p) > 1 = \int x dG(x)$. Hence $F(\cdot)$ does not first-order stochastically dominate $G(\cdot)$, but the mean of $F(\cdot)$ is larger than that of $G(\cdot)$.

6.D.3 Any elementary increase in risk from a distribution $F(\cdot)$ is a mean-preserving spread of $F(\cdot)$. In Example 6.D.2, we saw that any mean-preserving spread of $F(\cdot)$ is second-order stochastically dominated by $F(\cdot)$. Hence the assertion follows.

6.D.4 Let $L = (p_1, p_2, p_3)$ and $L' = (p'_1, p'_2, p'_3)$ be two lotteries.

(a) By a direct calculation, the means of L and L' are $2 - p_1 + p_3$ and $2 - p'_1 + p'_3$. Thus the two lotteries have an equal mean if and only if $p_1 - p_3 = p'_1 - p'_3$. Hence they have an equal mean if and only if they are both on a segment that is parallel to the segment connecting the \$2-vertex and the middle point of the (\$1,\$3)-face, as depicted below:

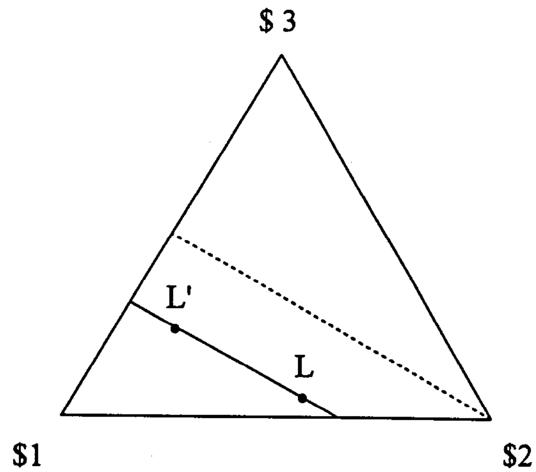


Figure 6.D.4(a)

(b) If the decision-maker exhibits risk aversion, then he prefers getting $2 with probability one to the lottery yielding $1 with probability 1/2 and $3 with probability 1/2. Hence the indifference lines are steeper than the segment connecting the $2-vertex and the middle point of the ($1,$3)-face. Hence, when L and L' have an equal mean, L is preferred to L' if and only if L is located on the right of L'. Therefore, L second-order stochastically dominates L' if and only if L is located on the right of L', as depicted in the figure below:

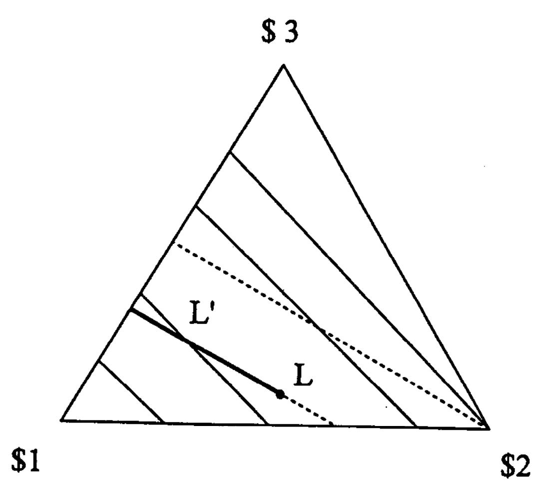


Figure 6.D.4(b)

(c) The distribution of L' is a mean preserving spread of that of L if and only if they are both on a segment that is parallel to the segment connecting the $2-vertex and the middle point of the ($1,$3)-face, and L' is closer to the ($1,$3)-face than L. This is depicted below:

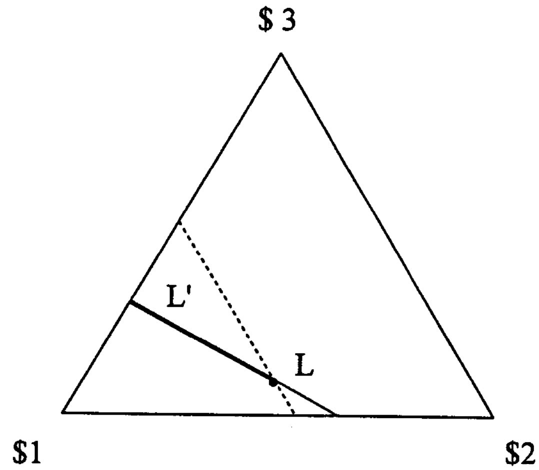


Figure 6.D.4(c)

(d) Inequality (6.D.1) holds if and only if $p_1' \geq p_1$ and $p_1' + (p_1' + p_2') \geq p_1 + (p_1 + p_2)$. But, since L and L' are assumed to have an equal mean, $p_1' - p_1 = p_3' - p_3$ and hence these two inequalities are equivalent to $p_1' \geq p_1$ alone. Thus, (6.D.1) holds if and only if L is located in the right of L', as depicted below:

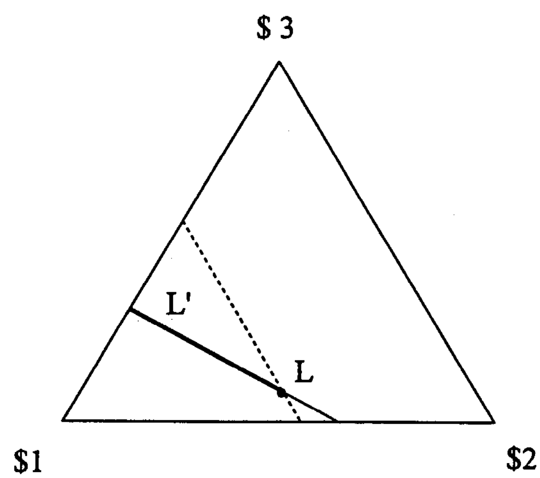


Figure 6.D.4(d)

6.E.1 Denote by $R(x,x')$ the expected regret associated with lottery x relative to x', and similarly for the other lotteries. A direct calculation yields:

$$R(x,x') = 2/3,\ R(x',x) = \sqrt{3}/3,$$

$$R(x',x'') = (\sqrt{2} + 1)/3,\ R(x'',x') = \sqrt{5}/3,$$

$$R(x'',x) = (\sqrt{2} + 1)/3,\ R(x,x'') = \sqrt{2}/3.$$

Thus, x' is preferred to x, x" is preferred to x', but x is preferred to x" .

6.E.2 **(a)** Denote the probability of state s by π_s and the expected utility from the contingent commodity vector (x_1,x_2) by $U(x_1,x_2)$, then $U(x_1,x_2) = \pi_1 u(x_1) + \pi_2(1 - \pi)u(x_2)$. Since $u(\cdot)$ is concave by the assumption of risk aversion, $U(\cdot)$ is also concave. Thus the preference ordering on (x_1,x_2) is convex.

(b) According to Exercise 6.C.5(a), the concavity of $U(\cdot)$ implies the risk aversion for the lotteries on (x_1,x_2).

(c) By the additive separability of $U(\cdot)$ and Exercise 3.G.4(c), both x_1 and x_2 are normal goods.

6.E.3 Since $g^*(s) = 1 + \alpha(g(s) - 1)$ for every s, we have

$$g^*(s) > g(s) \text{ if } g(s) < 1;$$

$$g^*(s) = g(s) \text{ if } g(s) = 1;$$

$$g^*(s) < g(s) \text{ if } g(s) > 1.$$

Thus $G^*(x) \le G(x)$ for every $x < 1$ and $G^*(x) \ge G(x)$ for every $x > 1$. Since $G(\cdot)$ and $G^*(\cdot)$ are continuous from the right, we have $G^*(1) \ge G(1)$. Hence property (6.D.2) holds and thus $G^*(\cdot)$ second-order stochastically dominates $G(\cdot)$ weakly. (If $g(s) \ne 1$ for some s, then $G^*(x) < G(x)$ for some $x < 1$ and

$G^*(x) > G(x)$ for some $x > 1$. Hence, in this case, $G^*(\cdot)$ second-order stochastically dominates $G(\cdot)$ strictly.)

6.F.1 We shall first prove the uniqueness of the utility function on money up to origin and scale. Suppose that two utility function $u(\cdot)$ and $\hat{u}(\cdot)$ satisfy the condition of the theorem. Since the state preferences $\succsim_s$ are represented by both $\int(\pi_s u(x_s) + \beta_s)dF_s(x_s)$ and $\int(\hat{\pi}_s \hat{u}(x_s) + \hat{\beta}_s)dF_s(x_s)$, by applying Proposition 6.B.2 to the set of all the lotteries in some state s, we know that $\pi_s u(\cdot) + \beta_s$ and $\hat{\pi}_s \hat{u}(\cdot) + \hat{\beta}_s$ are the same up to origin and scale. Hence so are $u(\cdot)$ and $\hat{u}(\cdot)$.

It remains to verify the uniqueness of subjective probability. Suppose that both $\sum_s \pi_s(\int u(x_s)dF_s(x_s))$ and $\sum_s \hat{\pi}_s(\int \hat{u}(x_s)dF_s(x_s))$ represents the same preference relation on $\mathcal{L}$. Now that we have shown that $u(\cdot)$ and $\hat{u}(\cdot)$ are the same up to origin and scale, without loss of generality, we can assume that $u(\cdot) = \hat{u}(\cdot)$. We can normalize $u(\cdot)$ so that $u(0) = 0$ and $u(1) = 1$. Note here that if a distribution function $F_s(\cdot)$ puts probability p_s on 1 and probability $1 - p_s$ on 0, then the expected utility is p_s. Thus, by choosing p_s suitably for each s, any point in $[0,1]^S$ can be represented in the form

$$(\int u(x_1)dF_1(x_1),\dots,\int u(x_S)dF_S(x_S)).$$

Hence, if $(\pi_1,\dots,\pi_S) \neq (\hat{\pi}_1,\dots,\hat{\pi}_S)$, then there would exist $(F_1,\dots,F_S) \in \mathcal{L}$ and $(F'_1,\dots,F'_S) \in \mathcal{L}$ such that

$$\sum_s \pi_s(\int u_s(x_s)dF_s(x_s)) > \sum_s \pi_s(\int u_s(x_s)dF'_s(x_s)),$$

$$\sum_s \hat{\pi}_s(\int u_s(x_s)dF_s(x_s)) < \sum_s \hat{\pi}_s(\int u_s(x_s)dF'_s(x_s)).$$

This contradicts the assumption that they represent the same preference. Thus $(\pi_1,\dots,\pi_S) = (\hat{\pi}_1,\dots,\hat{\pi}_S)$.

6.F.3 **(a)** If $P = \{\pi\}$, then $U_W(H) = \pi$ and $U_B(H) = 1 - \pi$. Hence they are determined from the expected utility $\pi u(1000) + (1 - \pi)u(0)$. Moreover, $U_W(R) > U_W(H)$ if and only if $0.49 > \pi$. But this is equivalent to $0.51 < 1 - \pi$, which is, in turn, equivalent to $U_B(R) < U_B(H)$.

(b) We have $U_W(R) > U_W(H)$ if and only if $0.49 > \text{Min } P$. We have $U_B(R) > U_B(H)$ if and only if $0.51 > \text{Min}\{1 - \pi: \pi \in P\}$, which is equivalent to $0.49 < \text{Max } P$. Hence $\text{Min } P < 0.49 < \text{Max } P$ if and only if $U_W(R) > U_W(H)$ and $U_B(R) > U_B(H)$.

CHAPTER 7

7.C.1

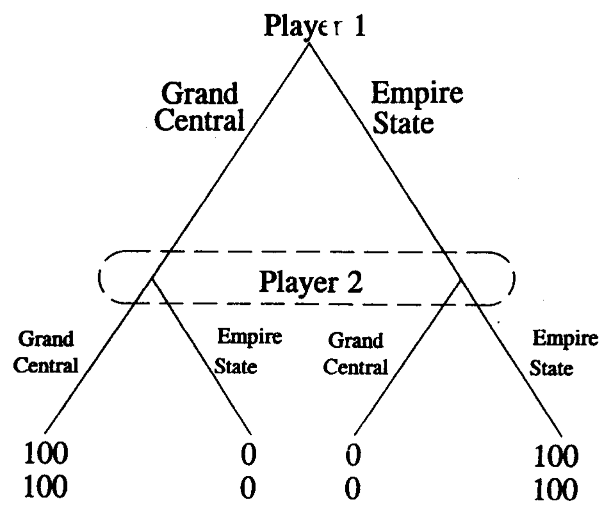


Figure 7.C.1

7.D.1 Player i has $M_1 \times M_2 \times M_3 \times \cdots \times M_N$ strategies in this game. □

7.D.2

		Player 2: H	Player 2: T
Player 1	H	-1 , 1	1 , -1
	T	1 , -1	-1 , 1

Figure 7.D.2

7.E.1 **(a)** In order to specify a strategy for player 1, we need to determine his moves in all of the three information sets in which he moves. Thus a typical strategy for player 1 can be written as a triple. The set of strategies for player 1 are:

S_1 = { (L, x, x), (L, x, y), (L, y, x), (L, y, y), (M, x, x), (M, x, y), (M, y, x), (M, y, y), (R, x, x), (R, x, y), (R, y, x), (R, y, y) }

If player 1 uses strategy (L, x, y), he plays L at the root of the game, x in his information set following his move M (we refer to this information set as "Information Set 2") and y in his information set following his move R We refer to this information set as "Information set 3").

Similarly player 2's strategy specify her move at her information set (we refer tho this information set as "Information Set 1"). Thus, $S_2 = \{(l),(r)\}$.

(b) A behavior strategy for player 1 consists of a randomization of his possible moves at each information set in which he has to move. Suppose that at the root, player 1 plays L, M, and R with probabilities of p_1, p_2 and p_3 respectively ($p_1+p_2+p_3=1$); at information set 2, player 1 plays x, y with probabilities of q_1 and q_2 respectively ($q_1+q_2=1$); at information set 3, player 1 plays x, y with probabilities of s_1 and s_2 respectively.

Assume that player 2 plays l and r with probabilities $\sigma(l)$ and $\sigma(r)$ respectively ($\sigma(l)+\sigma(r)=1$). Thus, if player 1 is using the above behavioral strategy and player 2 is using this mixed strategy, the probability that we reach each terminal node will be:

$\Pr(T_0) = p_1$; $\Pr(T_1) = p_2\ \sigma(l)\ q_1$; $\Pr(T_2) = p_2\ \sigma(l)\ q_2$; $\Pr(T_3) = p_2\ \sigma(r)\ q_1$;
$\Pr(T_4) = p_2\ \sigma(r)\ q_2$; $\Pr(T_6) = p_3\ \sigma(l)\ r_2$; $\Pr(T_7) = p_3\ \sigma(r)\ r_1$;
$\Pr(T_8) = p_3\ \sigma(r)\ r_2$.

Now the following mixed strategy for player 1 is realization equivalent to the above behavior strategy:

(L, x, x) with probability p_1, (M, x, x) with probability p_2q_1,
(M, y, x) with probability p_2q_2, (R, x, x) with probability p_3r_1,
(R, x, y) with probability p_3r_2. [Note: $p_1 + p_2q_1 + p_2q_2 + p_3r_1 + p_3r_2 = p_1 + p_2(q_1 + q_2) + p_3(r_1 + r_2) = p_1 + p_2\ 1 + p_3\ 1 = 1$]

If player 1 is using the above mixed strategy and player 2 is using the mixed strategy σ, the probability that we reach each terminal node will be the same as shown before for the behavior strategy. Therefore, the above mixed strategy is realization equivalent to the behavior strategy.

(c) Suppose that player 1 uses the following mixed strategy:

(L, x, x) with probability p_1 ; (L, x, y) with probability p_2,
(L, y, x) with probability p_3 ; (L, y, y) with probability p_4,
(M, x, x) with probability p_5 ; (M, x, y) with probability p_6,
(M, y, x) with probability p_7 ; (M, y, y) with probability p_8,
(R, x, x) with probability p_9 ; (R, x, y) with probability p_{10},
(R, y, x) with probability p_{11} ; (R, y, y) with probability p_{12}.
[$p_i \geq 0$ for all i and $\sum p_i = 1$]

If Player 2 uses the mixed strategy σ, the probability that we reach each terminal node will be: $\Pr(T_0) = p_1 + p_2 + p_3 + p_4$, $\Pr(T_1) = (p_5 + p_6)\,\sigma(l)$, $\Pr(T_2) = (p_7 + p_8)\sigma(l)$, $\Pr(T_3) = (p_5 + p_6)\,\sigma(r)$, $\Pr(T_4) = (p_7 + p_8)\,\sigma(r)$, $\Pr(T_5) = (p_9 + p_{10})\,\sigma(l)$, $\Pr(T_6) = (p_{11} + p_{12})\,\sigma(l)$, $\Pr(T_7) = (p_9 + p_{10})\,\sigma(r)$, $\Pr(T_8) = (p_{11} + p_{12})\,\sigma(r)$.

The following behavioral strategy for player 1 is realization equivalent: At the root of the game, player 1 plays L, M, R with probabilities of $(p_1 + p_2 + p_3 + p_4)$, $(p_5 + p_6 + p_7 + p_8)$ and $(p_9 + p_{10} + p_{11} + p_{12})$ respectively; at information set 2, player 1 plays x, y with probabilities of $(p_5 + p_6)/(p_5 + p_6 + p_7 + p_8)$ and $(p_7 + p_8)/(p_5 + p_6 + p_7 + p_8)$ respectively; at information set 3, player 1 plays x, y with probabilities of $(p_9 + p_{10})/(p_9 + p_{10} + p_{11} + p_{12})$ and $(p_{11} + p_{12})/(p_9 + p_{10} + p_{11} + p_{12})$ respectively.

(d) Note that if player 1 reaches his (only) information set after player 2 moves, he will not remember whether he chose M or R. Thus, the game is not of perfect recall.

The result of part **(b)** still holds: there exists a mixed strategy for player 1 which is realization equivalent to any behavior strategy. Suppose player 1 uses the following behavior strategy:

At information set 1, player 1 plays L, M, R with probabilities of p_1, p_2 and p_3 respectively; at information set 2, player 1 plays x, y with probabilities of q_1 and q_2 respectively. If player 2 is using the mixed strategy σ, then the probability that we reach each terminal node will be:

$\Pr(T_0) = p_1$, $\Pr(T_1) = p_2\ \sigma(l)\ q_1$, $\Pr(T_2) = p_2\ \sigma(l)\ q_2$, $\Pr(T_3) = p_2\ \sigma(r)\ q_1$, $\Pr(T_4) = p_2\ \sigma(r)\ q_2$, $\Pr(T_5) = p_3\ \sigma(l)\ q_1$, $\Pr(T_6) = p_3\ \sigma(l)\ q_2$, $\Pr(T_7) = p_3\ \sigma(r)\ q_1$, $\Pr(T_8) = p_3\ \sigma(r)\ q_2$.

The following mixed strategy for player 1 is realization equivalent:

(L, x) with probability p_1, (M, x) with probability p_2q_1,
(M, y) with probability p_2q_2, (R, x) with probability p_3q_1, (M, y) with probability p_3q_2.

However, there does not always exist a behavior strategy that is realization equivalent to a mixed strategy. Consider the following example. Player 1 uses the mixed strategy playing (M, x) and (R, y) both with probability 1/2. Player 2 uses the pure strategy (l). Suppose there exist a behavior strategy for player 1 which is realization equivalent to the mixed strategy: at the root of the game, player 1 plays L, M, R with probabilities of p_1, p_2 and p_3 respectively; at his information set after player 2 moves, player 1 plays x, y with probabilities of q_1 and q_2 respectively. The mixed strategy generates the following distribution over the terminal nodes:

$\Pr(T_1) = \Pr(T_6) = 1/2$

$\Pr(T_0) = \Pr(T_2) = \Pr(T_3) = \Pr(T_4) = \Pr(T_5) = \Pr(T_7) = \Pr(T_8) = 0$ The behavior strategy generates:

$\Pr(T_3) = \Pr(T_4) = \Pr(T_7) = \Pr(T_8) = 0$

$\Pr(T_0) = p_1$, $\Pr(T_1) = p_2q_1$, $\Pr(T_2) = p_2q_2$, $\Pr(T_5) = p_3q_1$, $\Pr(T_6) = p_3q_2$.

in order for these distributions to be equivalent, we need: $\Pr(T_1) = p_2q_1 = 1/2 \Rightarrow p_2$ and $q_1 \neq 0$, $\Pr(T_2) = p_2q_2 = 0 \Rightarrow q_2 = 0$ since $p_2 \neq 0$, $\Pr(T_6) = p_3q_2 =$

1/2 which cannot hold since $q_2 = 0$, a contradiction. There exists no behavior strategy that is realization equivalent to the above mixed strategy.

In a game that is not of perfect recall the following holds:

- for any behavior strategy there exists a mixed strategy that is realization equivalent,
- not for all mixed strategies does there exist a realization equivalent behavior strategy. [Note: for a general proof of these results refer to Fudenberg/Tirole (1991) *Game Theory*. MIT press, p. 87]

CHAPTER 8

8.B.1 Firm i chooses h_i to maximize $\alpha\Sigma_j h_j + \beta(\Pi_j h_j) - w_i(h_i)^2$. The F.O.C. is: $\alpha + \beta(\prod_{j\neq i} h_j) - 2w_i h_i = 0$. The best response function for firm i is therefore: $h_i = [\alpha + \beta(\prod_{j\neq i} h_j)]\frac{1}{2w_i}$. Therefore firm i has a strictly dominant strategy iff $\beta = 0$, i.e., if the best response function of i is not dependent on the action of the other firms. If $\beta = 0$, firm i's strictly dominant strategy is $h_i = \frac{\alpha}{2w_i}$.

8.B.2 (a) Suppose $s_i^1 \in S_i$ and $s_i^2 \in S_i$ are two weakly dominant strategies for player i. This implies that $u_i(s_i^1, s_{-i}) \geq u_i(s_i^*, s_{-i})\ \forall\ s_i^* \in S_i$ and $\forall\ s_{-i} \in S_{-i}$, and $u_i(s_i^2, s_{-i}) \geq u_i(s_i^*, s_{-i})\ \forall\ s_i^* \in S_i$ and $\forall\ s_{-i} \in S_{-i}$. In particular, $u_i(s_i^1, s_{-i}) \geq u_i(s_i^2, s_{-i})$ and $u_i(s_i^2, s_{-i}) \geq u_i(s_i^1, s_{-i})\ \forall\ s_{-i} \in S_{-i}$. Therefore, $u_i(s_i^1, s_{-i}) = u_i(s_i^2, s_{-i})\ \forall\ s_{-i} \in S_{-i}$.

(b)

		Player 2 L	Player 2 R
Player 1	U	1 , 4	2 , 5
Player 1	D	1 , 2	2 , 3

Figure 8.B.2

Both of player 1's strategies (U) and (D) are weakly dominant. However, player 2 prefers that player 1 uses strategy (U).

8.B.3 Suppose not. Assume bidder i bids $b_i > v_i$. Then if some other bidder bids something larger than b_i, bidder i is just as well of as if he would have bid v_i. If all other players bid lower than v_i, then bidder i obtains the object and pays the amount of the second highest bid. If the second highest bid is $b_j < v_i$, this results in the same payoff for player i as if he bid v_i. However, suppose that the second highest bid of the other is $b_j > v_i$. Then, by bidding b_i bidder i will win the object and obtain a negative payoff. By bidding v_i he will not win the object and obtain a payoff of zero. Therefore, bidding $b_i > v_i$ is weakly dominated by bidding v_i.

Suppose bidder i bids $b_i < v_i$. Then if all other bidders bid something smaller than b_i, bidder i is just as well of as if he would have bid v_i. He will win the object and pay the the second highest bid. If some other player bids higher than v_i, then bidder i does not win the object regardless whether he bids b_i or v_i. However, suppose that nobody bids higher than v_i and the highest bid of the other players is b_j with $b_i < b_j < v_i$. Then by bidding b_i bidder i will not win the object, therefore getting a payoff of 0. By bidding v_i, he would win the object, pay $b_j < v_i$, and thus obtain a payoff of $v_i - b_j > 0$. Therefore, bidding $b_i < v_i$ is weakly dominated by bidding v_i. This argument implies that bidding v_i is a weakly dominant strategy.

8.B.4 Call the set of strategies for player i that remain after N rounds of deletion of strictly dominated strategies Σ_i^N. Suppose $s_i \in \Sigma_i^N$ is a strictly dominated strategy given the strategies Σ^N_{-i} of the other players Therefore, there exists a strategy $s_i^* \in \Sigma_i^N$, which is not a strictly dominated strategy given Σ^N_{-i}, which strictly dominates s_i. Suppose further that s_i will not be deleted in the N+1 round.

Since s_i was strictly dominated by s_i^* given Σ^N_{-i}, it will still be strictly dominated by s_i^* given $\Sigma^{N+1}_{-i} \subseteq \Sigma^N_{-i}$, and $s_i^* \in \Sigma_i^{N+1}$ (with the given assumptions). Thus the strategy s_i will be deleted in the next round. (Note: If s_i is only weakly dominated by s_i^* given Σ^N_{-i}, then s_i may no longer be weakly dominated given Σ^{N+1}_{-i} since Σ^{N+1}_{-i} may no longer include the strategies of the opponent relative to which some other strategy of player i will strictly be better. Thus the order of deletion does matter for the set of strategies surviving a process of iterated deletion of weakly dominated strategies).

8.B.5 **(a)** Suppose, player j produces q_j. Player i's best response can be calculated by maximizing (this is symmetric for both players):

$$\max\ [a - b(q_i + q_j) - c]q_i$$

which yields the F.O.C: $[a - b(2q_i + q_j) - c] = 0$, so the best response is: $b_i(q_j) = \frac{a-c}{2b} - \frac{q_j}{2}$.

Now, since $q_1 \geq 0$, $q_2 \leq (a-c)/2b$ (all other strategies would be strictly dominated by $q_2 = (a-c)/2b$). Therefore, since $q_2 \leq (a-c)/2b$, we have that $q_1 \geq (a-c)/2b - (a-c)/4b = (a-c)/4b$. Thus, since $q_1 \geq (a-c)/4b$, then $q_2 \leq (a-c)/2b - (a-c)/8b = 3(a-c)/8b$. Continuing in this fashion we will obtain: $q = (a-c)/2b - q/2$. Thus, after successive elimination of strictly dominated strategies, $q_1 = q_2 = (a-c)/3b$.

(b) Suppose, player j produces q_j and player h produces q_h. Player i's best response can be calculated by maximizing $[a - b(q_i + q_j + q_h) - c]q_i$, which yields the F.O.C $[a - b(2q_i + q_j + q_h) - c] = 0$, implying the best response: $b_i(q_j,q_h) = (a-c)/2b - (q_j + q_h)/2$.

Now, since $q_2, q_3 \geq 0$, $q_1 \leq (a-c)/2b$ (all other strategies would strictly be dominated by $q_1 = (a-c)/2b$). Thus, since $q_1 \leq (a-c)/2b$ and similarly $q_3 \le (a-c)/2b$, we have $q_2 \ge (a-c)/2b - [2(a-c)/2b]/2 = 0$. Therefore, successive elimination of strictly dominated strategies, implies that $q_1, q_2, q_3 \ge 0$ and $q_1, q_2, q_3 \le (a-c)/2b$. However, a unique prediction cannot be obtained.

8.B.6 Suppose s_i^* is strictly dominated by the strategy σ_i^*. Suppose further that σ_i is a mixed strategy in which s_i^* is played with strictly positive probability $\sigma_i(s_i^*)>0$. We claim that σ_i is strictly dominated by the mixed strategy σ_i', which is equivalent to σ_i except that instead of playing s_i^* with probability $\sigma_i(s_i^*)$ it plays σ_i^* with probability $\sigma_i(s_i^*)$. This follows since:

$$\begin{aligned} u_i(\sigma_i, \sigma_{-i}) &= \Sigma\, \sigma_i(s_i)\, u_i(s_i, \sigma_{-i}) \\ &= \sum_{s_i \neq s_i^*} \sigma_i(s_i)\, u_i(s_i, \sigma_{-i}) + \sigma_i(s_i^*)\, u_i(s_i^*, \sigma_{-i}) \\ &< \sum_{s_i \neq s_i^*} \sigma_i(s_i)\, u_i(s_i, \sigma_{-i}) + \sigma_i(s_i^*)\, u_i(\sigma_i^*, \sigma_{-i}) \\ &= u_i(\sigma_i', \sigma_{-i}) \quad \text{for all } \sigma_{-i} \in \Delta(S_{-i}). \end{aligned}$$

8.B.7 Suppose in negation that σ_i is a strictly dominant mixed strategy of player i, and suppose, $s_i^1, \ldots, s_i^N$ are the pure strategies that are played with positive probability in the mixed strategy σ_i. Since σ_i is a strictly dominant strategy: $u_i(\sigma_i, s_{-i}) > u_i(s_i^*, s_{-i})$ $\forall\, s_i^* \in S_i$ and $\forall\, s_{-i} \in S_{-i}$. In particular, $u_i(\sigma_i,s_{-i}) > u_i(s_i^j,s_{-i})$ $\forall\, j = 1,\ldots, N$. This implies that

$$u_i(\sigma_i,s_{-i}) > \sum_{j=1}^{N} [\sigma_i(s_i^j)\, u_i(s_i^j,s_{-i})] = u_i(\sigma_i,s_{-i}) \text{ , a contradiction.}$$

8.C.1 Notice that the elimination of strategies that are never a best-response is more demanding than strictly dominated strategy elimination. Thus, in every round of elimination, the deletion of never a best response deletes more strategies than the deletion of strictly dominated strategies.

Therefore, if the elimination of strictly dominated strategies yields a unique prediction in a game, then the elimination of strategies that are never a best-response cannot yield more than one strategy. Since a rationalizable strategy always exist, the elimination of strategies that are never a best-response will then also yield a unique prediction.

If the unique rationalizable strategy is not the unique prediction after elimination of strictly dominated strategies, then there exist a round of elimination in which this unique rationalizable strategy was strictly dominated. However, if this strategy was strictly dominated it was also never a best-response. This contradicts the assumption that the strategy is rationalizable. Therefore, both procedures must yield the same prediction.

8.C.2 Call the set of strategies for player i that remain after N rounds of deletion of never best-response strategies Σ_i^N. Suppose s_i is never a best-response to any strategy in Σ^N_{-i}. Suppose further that s_i will not be deleted in the N+1 round. Since s_i was never a best response to a strategy in Σ^N_{-i}, it will clearly not be a best-response to a strategy in $\Sigma^{N+1}_{-i} \subseteq \Sigma^N_{-i}$. Thus this strategy will be deleted in the next round.

8.C.3 Suppose that s_1 is a pure strategy of player 1 that is never a best response for any mixed strategy of player 2. Suppose in negation that s_1 is not strictly dominated. Construct the following correspondence for any $\sigma_i \in \Delta(S_i)$ for i = 1, 2:

$$(\sigma_1, \sigma_2) \to \{\hat{\sigma}_1 \mid \hat{\sigma}_1 \in \operatorname{argmax} g_1(\sigma_1,\sigma_2)\} \times \{\hat{\sigma}_2 \mid g_1(s_1, \hat{\sigma}_2) \geq g_1(\sigma_1, \hat{\sigma}_2)\}.$$

The first part of this correspondence is the best response function for player 1 and therefore satisfies all the conditions of the Kakutani fixed point theorem. The second part of the correspondence is the set of mixed strategies of player 2, for which s_1 is not strictly dominated (it is a non-empty set since s_1 is not a strictly dominated strategy. i.e., it is not strictly dominated by σ_1). Therefore, the second part of the correspondence is convex valued and upper hemicontinuous due to the usual assumptions. Thus, by Kakutani's theorem there exists a fixed point (σ_1^*, σ_2^*) of this correspondence such that $g_1(s_1, \sigma_2^*) \geq g_1(\sigma_1^*, \sigma_2^*)$ from the second part of the correspondence, and $g_1(\sigma_1^*, \sigma_2^*) \geq g_1(\sigma_1, \sigma_2^*)$ for all $\sigma_1 \in \Delta(S_1)$. Therefore, $g_1(s_1, \sigma_2^*) \geq g_1(\sigma_1, \sigma_2^*)$ for all $\sigma_1 \in \Delta(S_1)$, which contradicts the assumption that s_1 is a pure strategy of player 1 that is never a best response for any mixed strategy of player 2. Therefore, if s_1 is a pure strategy of player 1 that is never a best response for any mixed strategy of player 2, then s_1 is strictly dominated by some mixed strategy of player 1.

8.C.4 [First Printing Errata: a typo appears in the lower left box of the payoff matrix. Player 1's payoff should be $\pi + 4\varepsilon$ and <u>not</u> $\eta + 4\varepsilon$.]

For the continuation of this answer, a strategy for player 2 is to play u with probability α and D with probability $1-\alpha$, and for player 3 is to play l with probability β and r with probability $1-\beta$. Denote by P_A the expected payoff of player 1 when action $A \in \{L,M,R\}$ is taken given α and β. Direct calculation and simple algebra yield:

$$P_M = \pi + \left(\frac{3\alpha + 3\beta}{2} - 3\alpha\beta - 1 \right)\eta$$

$$P_L = \pi + (2\beta - 1)\varepsilon$$

$$P_R = \pi + (1 - 2\beta)\varepsilon$$

(a) To show that M is never a best response to any pair of strategies of players 2 and 3, (α,β), we have three cases:

<u>Case 1:</u> $\beta > 1/2$

Note that in this case $\frac{\partial P_M}{\partial \alpha} = \eta[3/2 - 3\beta] < 0$. Thus the highest payoff for player 1 if he plays M is obtained when $\alpha = 0$, and his payoff will be $P_M(\alpha{=}0) = \pi + \eta[\frac{3}{2}\beta - 1] < \pi + 4\varepsilon[\frac{3}{2}\beta - 1] < \pi + 4\varepsilon[2\beta - 1] = P_L$. Further note that P_L is independent of α, so that these inequalities hold for all α. Therefore, M cannot be a best response in this case.

Case 2: $\beta < 1/2$

Now, $\frac{\partial P_M}{\partial \alpha} > 0$, the highest payoff for player 1 if he plays M is obtained when $\alpha = 1$, and his payoff is $P_M(\alpha{=}1) = \pi + \eta[\frac{3}{2} + \frac{3}{2}\beta - 3\beta - 1] = \pi + \eta[\frac{1}{2} - \frac{3}{2}\beta] < \pi + \eta[\frac{1}{2} - \frac{3}{2}\beta + \frac{1}{2} - \beta] < \pi + 4\varepsilon[1 - 2\beta] = P_R$. Further note that P_R is independent of α, so that these inequalities hold for all α. Therefore, M cannot be a best response in this case.

Case 3: $\beta = 1/2$

In this case $P_M = \pi - \frac{\eta}{4} < \pi = P_R = P_L$. This concludes that M can never be a best response.

(b) Suppose in negation that there exists a mixed strategy, in which player 1 plays R with probability γ and L with probability $1-\gamma$, that strictly dominates M.

Case 1: $\gamma \leq 1/2$.

If $\beta = 0$ and $\alpha = 1$ then $P_M = \pi + \eta/2 > \pi$. The mixed strategy will give a payoff of $\pi - 4\varepsilon(1-2\gamma) \leq \pi$. Therefore, M cannot be a strictly dominated by the mixed strategy in this case.

Case 2: $\gamma > 1/2$.

If $\beta = 1$ and $\alpha = 0$ then $P_M = \pi + \eta/2 > \pi$. The mixed strategy will give a payoff of $\pi - 4\varepsilon(2\gamma-1) \leq \pi$. Therefore, M cannot be a strictly dominated by the mixed strategy in this case. This implies a contradiction, so that M cannot be strictly dominated.

(c) Suppose players correlate in the following way: Players 2 and 3 play

(U, r) with probability 1/2 and (D, l) with probability 1/2. Any mixed strategy for player 1 involving only L and R will give him a payoff of π. However, playing M will yield him a payoff of $\pi + \eta/2$. Thus M is a best-response to the above correlated strategy of player 2 and 3.

8.D.1 We know already from section 8.C that a_4 and b_4 are not rationalizable strategies. Thus, these strategies cannot be played with positive probability in a mixed strategy Nash equilibrium. Suppose that there exists a mixed strategy equilibrium in which a_1 and a_3 are both played with a strictly positive probability. Then the expected payoff from playing either one of them has to be equal (see exercise 8.D.2). This implies that the probability that player 2 plays b_1 has to be equal to the probability that he plays b_3. Now, suppose that player 2 plays b_1 and b_3 with probability α and b_2 with probability $1-2\alpha$. The expected payoff for player 1 obtained by playing either a_1 or a_3 equals: $7\alpha + (1-2\alpha)2$. The expected payoff for player 1 when playing a_2 equals: $5\alpha + 5\alpha + (1-2\alpha)3 = 10\alpha + (1-2\alpha)3 > 7\alpha + (1-2\alpha)2$. Therefore, in a mixed strategy equilibrium a_1 and a_3 cannot both be played with positive probability since playing a_2 would give the player a larger payoff.

Suppose, there exists a mixed strategy equilibrium in which player 1 plays a_1 and a_2 with strictly positive probability. Clearly, player 2's best response to this strategy of player 1 does not involve playing b_3 with strictly positive probability (given the strategy of player 1, playing b_2 is strictly better for player 2). Thus player 2 will play b_1 with probability β and b_2 with probability $1-\beta$. The payoff for player 1 from playing a_1 equals: $(1-\beta)2$, playing a_2 yields: $5\beta + (1-\beta)3 > (1-\beta)2$. Therefore, in a mixed strategy equilibrium a_1 and a_2 cannot both be played with strictly positive probability since playing a_2 is always better.

Similarly, it can be shown that there exists no mixed strategy equilibrium in which a_2 and a_3 are both played with strictly positive probability. Therefore, player 1 always plays a_2 in a Nash equilibrium. Player 2 will then play his best response b_2. Thus (a_2, b_2) being played with certainty is the unique mixed strategy equilibrium.

8.D.2 We will show that any Nash equilibrium (NE) must be in S^∞, the set of strategies which survive iterated strict dominance. Since it is assumed that this set contains one element, this will prove the required result.

Let $(s_1^*, s_2^*, \ldots, s_I^*)$ be a (mixed) NE and suppose in negation that it does not survive iterated strict dominance. Let i be the player whose strategy is first ruled out in the iterative process (say in the k^{th} round). Therefore, there exists σ_i and a_i such that $u_i(\sigma_i, s_{-i}) > u_i(a_i, s_{-i})\ \forall\ s_{-i} \in S_{-i}^{k-1}$, and a_i is played with positive probability $s_i^*(a_i)$. Since k is the first round at which any of the NE strategies, $(s_1^*, s_2^*, \ldots, s_I^*)$, are ruled out, we must have that $s_{-i}^* \in S_{-i}^{k-1}$. Hence, $u_i(\sigma_i, s_{-i}^*) > u_i(a_i, s_{-i}^*)$. Let the strategy s_i' be derived from s_i^* execpt that any probability of playing a_i is replaced by playing σ_i. We thus have that:

$$u_i(s_i', s_{-1}^*) = u_i(s_i^*, s_{-1}^*) + s_i^*(a_i)\cdot[u_i(\sigma_i, s_{-1}^*) - u_i(a_i, s_{-1}^*)] > u_i(s_i^*, s_{-1}^*)$$

which contradicts the assumption that $(s_1^*, s_2^*, \ldots, s_I^*)$ is a NE.

8.D.3 First of all, notice that the first auction bid is a simultaneous move game where a strategy for a player consists of a bid. Let b_1 be the bid of Player 1, and b_2 be the bid of Player 2.

(i) If $b_1 > b_2$, Player 1 gets the object and pays b_1 for it; Player 2 does not get the object. Thus, in this case:

$$\begin{cases} u_1(b_1,b_2)=v_1-b_1 & \text{[1's valuation of the object minus what he has to pay for it].} \\ u_2(b_1,b_2)=0 \end{cases}$$

(ii) Similarly, if $b_2 > b_1$: $\begin{cases} u_1(b_1,b_2)=0 \\ u_2(b_1,b_2)=v_2-b_2 \end{cases}$

(iii) If $b_1 = b_2$, each player gets the object with probability $\frac{1}{2}$:

$u_1(b_1 + b_2) = \frac{1}{2}(v_1 - b_1)+ \frac{1}{2}\cdot 0 = \frac{(v_1-b_1)}{2}$.

Similarly: $u_2(b_1, b_2) = \frac{(v_2 - b_2)}{2}$

Therefore, we have for $i,j \in \{1,2\}$, $i \neq j$:

$$u_i(b_1, b_2) = \begin{cases} 0, & b_i < b_j \\ \frac{1}{2}(v_i - b_i), & b_1 = b_2 \\ (v_i - b_i), & b_i > b_j \end{cases}$$

(a) We claim that no strategy for player 1 is strictly dominated. Suppose in negation that b_1 is strictly dominated by b_1', i.e., for any b_2: $u_1(b_1',b_2) > u_1(b_1,b_2)$. Take $b_2^* = \max\{b_1,b_1'\} + 1$, then: $b_2^* > b_1$, and $b_2^* > b_1'$. Hence, $u_1(b_1',b_2^*) = u_1(b_1,b_2^*) = 0$, a contradiction. Therefore, no strategy for player 1 is strictly dominated. Similarly, one can prove that no strategy for player 2 is strictly dominated, and thus no strategies are strictly dominated.

(b) We now claim that any strategy b_1 for Player 1 such that $b_1 > v_1$, is weakly dominated by v_1. Note that, if $b_1 > v_1$:

(i) If $b_2 < v_1$: $u_1(v_1, b_2) = v_1 - v_1 = 0$

$u_1(b_1, b_2) = v_1 - b_1 < 0 = u(v_1, b_2)$

(ii) If $b_2 = v_1$: $u_1(v_1, b_2) = \frac{1}{2}(v_1 - v_1) = 0$

$u_1(b_1, b_2) = v_1 - b_1 < 0 = u_1(v_1, b_2)$

(iii) If $v_1 < b_2 < b_1$: $u_1(v_1, b_2) = 0$

$u_1(b_1, b_2) = v_1 - b_1 < 0 = u_1(v_1, b_2)$

(iv) If $v_1 < b_2 = b_1$: $u_1(v_1, b_2) = 0$

$u_1(b_1,b_2) = 1/2(v_1 - b_1) < 0 = u_1(v_1,b_2)$

(v) If $b_2 > b_1$: $u_1(v_1, b_2) = 0$

$u_1(b_1, b_2) = 0 = u_1(v_1, b_2)$

Thus, in all cases, $u_1(v_1, b_2) \geq u_1(b_1, b_2)$, and in some cases strict inequality holds. Thus, $b_1 > v_1$ is weakly dominated by v_1. Similarly, any strategy b_2 for Player 2 such that $b_2 > v_2$ is weakly dominated by v_2.

Now suppose $v_1 > 2$. We claim that, in this case, $b_1 = 1$ weakly dominates $b'_1 = 0$. Observe that:

(i) If $b_2 = 0$: $u_1(1, 0) = v_1 - 1$

$u_1(0, 0) = \frac{1}{2} v_1$

Since $v_1 > 2$, $u_1(1,0) - u_1(0,0) = (v_1-1) - \frac{v_1}{2} = \frac{v_1}{2} - 1 > 0$

(ii) If $b_2 = 1$: $u_1(1, 1) = \frac{1}{2}(v_1 - 1) > 0$

$u_1(0,1) = 0$

(iii) If $b_2 > 1$: $u_1(1, b_2) = u_1(0,b_2) = 0$.

Thus, in all cases $u_1(1, b_2) \geq u_1(0,b_2)$, with strict inequality in some cases. Finally, suppose that $v_1 \in \{1,2\}$. We claim that, in this case, $b_1 = v_1 - 1$ weakly dominates $b_1 = v_1$:

(i) If $b_2 < v_1 - 1$: $u_1(v_1-1,b_2) = 1 > 0 = u_1(v_1,b_2)$

(ii) If $b_2 = v_1-1$: $u(v_1-1, b_2) = 1/2 > 0 = u_1(v_1,b_2)$

(iii) If $b_2 > v_1-1$: $u(v_1-1, b_2) = 0 = u_1(v_1,b_2)$

Thus, in all cases, $u(v_1-1, b_2) \geq u(v_1, b_2)$, with strict inequality in (i) and (ii). Similarly, it can be shown that:

-if $v_2 > 2$, $b_2 = 1$ weakly dominates $b'_2 = 0$

-if $v_2 \geq 1$, $b_2 = v_2-1$ weakly dominates $b'_2 = v_2$

(c) Define the best response correspondence for Player 1 as the set of maximizers of 1's utility, given the strategy for 2. We already have the expression for $u_1(b_1, b_2)$; in order to find out the best response correspondence, all we have to do is maximize this function. Denoting this best response by $R_1(b_2)$, direct maximization of $u(b_1, b_2)$ yields:

$$R_1(b_2) = \begin{cases} \{b_2+1\}, & \text{if } b_2 < v_1-2 \\ \{b_2, b_2+1\}, & \text{if } b_2 = v_1-2 \\ \{b_2\}, & \text{if } b_2 = v_1-1 \\ \{0,1,2,\ldots v_1\}, & \text{if } b_2 = v_1 \\ \{0,1,2,\ldots,b_2-1) & \text{if } b_2 > v_1 \end{cases}$$

Similarly, if $R_2(b_1)$ is the best response correspondence for Player 2:

$$R_2(b_1) = \begin{cases} \{b_1+1\}, & \text{if } b_1 < v_2-2 \\ \{b_1, b_1+1\}, & \text{if } b_1 = v_2-2 \\ \{b_1\}, & \text{if } b_1 = v_2-1 \\ \{0,1,\ldots,v_2\}, & \text{if } b_1 = v_2 \\ \{0,1,\ldots b_1-1\}, & \text{if } b_1 > v_2 \end{cases}$$

A Nash equilibrium is a pair (b_1^*, b_2^*) where $b_1^* \in R_1(b_2^*)$, and $b_2^* \in R_2(b_1^*)$. It can be verified that a NE always exists, and, for sufficiently large values of v_1 and v_2 Nash equilibria are not unique.

More explicitly, it can be shown that, for $v_1, v_2 \geq 2$ the following are Nash equilibria:

(i) If $v_1 = v_2$: (v_1, v_2), (v_1-1, v_2-1), (v_1-2, v_2-2)

(ii) If $v_1 = v_2+1$: (v_1-1, v_1-2), (v_1-2, v_1-2)

(iii) If $v_2 = v_1+1$: (v_2-2, v_2-1), (v_2-2, v_2-2)

(iv) If $v_1 > v_2+1$: (v_1-x, v_1-x-1), with $1 \leq x \leq v_1 - v_2$

Note: if $v_1 = v_2+2$: (v_1-2, v_1-2) is also a NE.

(v) If $v_2 > v_1+1$: (v_2-x-1, v_2-x), with $1 \leq x \leq v_2 - v_1$

Note: if $v_2 = v_1+2$: (v_2-2, v_2-2) is also a NE.

Thus, generally, uniqueness of NE does not hold, although existence does.

8.D.4 **(a)** If player i demands $y \geq 100$, then any strategy of player j with $x \geq 0$ is payoff equivalent. Therefore, there exists no strictly dominated strategy.

(b) Any strategy demanding more than \$100 is weakly dominated.

Case 1: player 2 demands $y \geq 100$. Then any strategy of player 1 with $x \geq 0$ is payoff equivalent. Case 2: player 2 demands $0 \leq y < 100$. Then, player 1 could demand $x = 100 - y$ and would obtain a payoff of $100 - y$. Demanding $x > 100$ will give player 1 a payoff of 0. Therefore, any strategy demanding more than \$100 is weakly dominated.

(c) Any pair $(x, 100 - x)$ with $100 \geq x \geq 0$ is a pure strategy Nash equilibrium of this game.

Proof: Suppose, player 1 demands x with $100 \geq x \geq 0$. If player 2 demands $y = 100 - x$, his payoff will equal $100 - x \geq 0$. If player 2 demands $y > 100 - x$, the demands sum to more than \$100 and both players get 0. If player 2 demands $0 \leq y < 100 - x$, he will obtain his demand and therefore be worse off than if he would have demanded $100 - x$. Thus, if player 1 demands x, player 2's best response is to demand $y = 100 - x$. Similarly if player 2 demands $100 - x$, player 1's best response is to demand x.

8.D.5 **(a)** Let x_1 be the location of Vendor 1 and x_2 be the location of Vendor 2. Thus, we can associate a strategy for Player i with $x_i \in [0,1]$. First, let us find out the payoff function for each of the vendors. Since the price of the ice cream is regulated, we can identify the profit of each vendor with the number of customers s/he gets. Suppose that $x_1 < x_2$. In this case, all consumers located to the left of (below) $\frac{x_1 + x_2}{2}$ will purchase from Vendor 1, while all customers located to the right of $\frac{x_1 + x_2}{2}$ will buy ice cream from Vendor 2. Thus:

$$u_1(x_1, x_2) = \frac{x_1 + x_2}{2} \qquad (= \text{length of } [0,\frac{x_1 + x_2}{2}])$$

$$u_2(x_1, x_2) = 1 - \frac{x_1 + x_2}{2} \qquad (= \text{length of } [\frac{x_1 + x_2}{2},1])$$

We can derive a similar result for $x_2 < x_1$:

$$u_1(x_1, x_2) = 1 - \frac{x_1 + x_2}{2}$$

$$u_2(x_1, x_2) = \frac{x_1 + x_2}{2}$$

Now, if $x_1 = x_2$, the vendors split the business so that $u_1(x_1,x_2) = u_2(x_1,x_2) = \frac{1}{2}$. Thus, summarizing:

$$u_1(x_1, x_2) = \begin{cases} \frac{x_1 + x_2}{2}, & x_1 < x_2 \\ \frac{1}{2}, & x_1 = x_2 \\ 1-\frac{x_1 + x_2}{2}, & x_1 > x_2 \end{cases}$$

$$u_2(x_1, x_2) = \begin{cases} 1 - \frac{x_1 + x_2}{2}, & x_1 < x_2 \\ \frac{1}{2}, & x_1 = x_2 \\ \frac{x_1 + x_2}{2}, & x_1 > x_2 \end{cases}$$

It is straightforward to check that $x_1 = x_2 = 1/2$ constitutes a NE (no firm can do better by deviating). To show uniqueness, suppose first that $x_1 = x_2 < 1/2$. Then any firm can do better by moving by $\varepsilon > 0$ to the right, since it will sell almost $1 - x_1 > 1/2$ units rather than 1/2 units. Similarly it can be shown that $x_1 = x_2 > 1/2$ does not constitute a NE. Suppose now that

$x_1 < x_2$. Then firm 1 can do better by moving to $x_2 - \varepsilon$, with $\varepsilon > 0$, therefore this could not have been a NE. Similarly it can be shown that $x_1 > x_2$ does not constitute a NE.

(b) Suppose that an equilibrium (x_1^*, x_2^*, x_3^*) exists. Suppose, first, that $x_1^* = x_2^* = x_3^*$. Then each firm will sell 1/3. But any firm can increase its sales by moving to the right (if $x_1^* = x_2^* = x_3^* < 1/2$) or the left (if $x_1^* = x_2^* = x_3^* \geq 1/2$), a contradiction. Suppose that two firms locate at the same point, let's say $x_1^* = x_2^*$. If $x_1^* = x_2^* < x_3^*$, then firm 3 can do better by moving to $x_1^* + \varepsilon$. If $x_1^* = x_2^* > x_3^*$, then firm 3 can do better by moving to $x_1^* - \varepsilon$, a contradiction. Finally, suppose that all 3 firms are located at different points. But then the firm that is located the farthest on the right will be able to increase its sales by moving to the left by $\varepsilon > 0$, a contradiction. Thus, there exists no pure strategy NE in this game.

8.D.6 <u>Case 1</u>: $u > w$ and $l > y$.

In this case player 1 always plays his dominant strategy a_1. Player 2 will play his best response to this strategy, i.e. if $v > m$ he will play b_1, if $v < m$ he will play b_2 and otherwise he will be indifferent.

<u>Case 2</u>: $u < w$ and $l < y$.

In this case player 1 always plays his dominant strategy a_2. Player 2 will play his best response to this strategy, i.e. if $x > z$ he will play b_1, if $x < z$ he will play b_2 and otherwise he will be indifferent.

<u>Case 3</u>: $v > m$ and $x > z$.

In this case player 2 always plays his dominant strategy b_1. Player 2 will play his best response to this strategy, i.e. if $u > w$ he will play a_1, if $u < w$ he will play a_2 and otherwise he will be indifferent.

<u>Case 4</u>: $v < m$ and $x < z$.

In this case player 2 always plays his dominant strategy b_2. Player 2 will play his best response to this strategy, i.e. if $l > y$ he will play a_1, if $l < y$ he will play a_2 and otherwise he will be indifferent.

Case 5: all other cases.

Suppose player 2 plays b_1 with probability α and b_2 with probability $1-\alpha$. Player 1's best response will be a mixed strategy if he obtains the same payoff from playing either of his strategies:

$$\alpha u + (1-\alpha)l = \alpha w + (1-\alpha)y, \quad \Rightarrow \quad \alpha = \frac{y - l}{u + y - l - w}.$$

Similarly suppose player 1 plays a_1 with probability β and a_2 with probability $1-\beta$. Player 2's best response will be a mixed strategy if he obtains the same payoff from playing either of his strategies:

$$\beta v + (1-\beta)x = \beta m + (1-\beta)z, \quad \Rightarrow \quad \beta = \frac{z - x}{v + z - x - m}..$$

Player 1 playing a_1 (a_2) with probability β ($1-\beta$) and player 2 playing b_1 (b_2) with probability α ($1-\alpha$) as defined above is a mixed strategy Nash equilibrium.

8.D.7 (a) Suppose $\underline{w}_i = u_i(\sigma_i^*, \sigma_{-i}^*)$. The following is true by definition:

$$\underline{v}_i = \min_{\sigma_{-i}} [\max_{\sigma_i} u_i(\sigma_i, \sigma_{-i})] \geq \min_{\sigma_{-i}} [u_i(\sigma_i^*, \sigma_{-i})] = u_i(\sigma_i^*, \sigma_{-i}^*) = \underline{w}_i.$$

(b) Suppose $(\sigma_i^*, \sigma_{-i}^*)$ is a mixed strategy Nash equilibrium. We have:

$u_i(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}^*)$ and $u_{-i}(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma_{-i}} u_{-i}(\sigma_i^*, \sigma_{-i})$. Since this is a zero-sum game we can rewrite the second equality:

$$-u_i(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma_{-i}} [-u_i(\sigma_i^*, \sigma_{-i})] \text{ or, } u_i(\sigma_i^*, \sigma_{-i}^*) = \min_{\sigma_{-i}} u_i(\sigma_i^*, \sigma_{-i}).$$

This yields: $\max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}^*) = u_i(\sigma_i^*, \sigma_{-i}^*) = \min_{\sigma_{-i}} u_i(\sigma_i^*, \sigma_{-i})$. Now,

$\min_{\sigma_{-i}} [\max_{\sigma_i} u_i(\sigma_i,\sigma_{-i})] \le \max_{\sigma_i} u_i(\sigma_i,\sigma^*_{-i}) = u_i(\sigma^*_i,\sigma^*_{-i})$, and also

$u_i(\sigma^*_i, \sigma^*_{-i}) = \min_{\sigma_{-i}} u_i(\sigma^*_i , \sigma_{-i}) \le \max_{\sigma_i} [\min_{\sigma_{-i}} u_i(\sigma_i , \sigma_{-i})]$. Taking these two

inequalities together we get:

$$\underline{v}_i = \min_{\sigma_{-i}} [\max_{\sigma_i} u_i(\sigma_i, \sigma_{-i})] \le u_i(\sigma^*_i, \sigma^*_{-i}) \le \max_{\sigma_i} [\min_{\sigma_{-i}} u_i(\sigma_i , \sigma_{-i})] = \underline{w}_i.$$

But we know from **(a)** above that $\underline{v}_i \ge \underline{w}_i$. Therefore, we must have:

$$\underline{v}_i = u_i(\sigma^*_i, \sigma^*_{-i}) = \underline{w}_i.$$

(c) Suppose $(\sigma^*_i, \sigma^*_{-i})$ and $(\sigma'_i, \sigma'_{-i})$ are mixed strategy Nash equilibria. We must therefore have that:

(i) $u_i(\sigma^*_i,\sigma^*_{-i}) \ge u_i(\sigma'_i,\sigma^*_{-i}) = -u_{-i}(\sigma'_i,\sigma^*_{-i}) \ge -u_{-i}(\sigma'_i,\sigma'_{-i}) = u_i(\sigma'_i,\sigma'_{-i})$

(ii) $u_i(\sigma^*_i,\sigma^*_{-i}) = -u_{-i}(\sigma^*_i,\sigma^*_{-i}) \le -u_{-i}(\sigma^*_i,\sigma'_{-i}) = u_i(\sigma^*_i,\sigma'_{-i}) \le u_i(\sigma'_i,\sigma'_{-i})$

(these inequalities follow from the properties of NE and from the zero-sum property). From part **(b)** we know that $u_i(\sigma'_i,\sigma'_{-i}) = u_i(\sigma^*_i, \sigma^*_{-i}) = \underline{v}_i = \underline{w}_i$. This, together with **(i)** and **(ii)** above yield:

$$\underline{v}_i = u_i(\sigma'_i,\sigma'_{-i}) \le u_i(\sigma'_i,\sigma^*_{-i}) \le u_i(\sigma^*_i,\sigma^*_{-i}) \le u_i(\sigma^*_i,\sigma'_{-i}) \le u_i(\sigma'_i,\sigma'_{-i}) = \underline{v}_i$$

Therefore, $u_i(\sigma'_i,\sigma^*_{-i}) = u_i(\sigma^*_i,\sigma^*_{-i}) = u_i(\sigma^*_i,\sigma'_{-i}) = u_i(\sigma'_i,\sigma'_{-i}) = \underline{v}_i$.

Since $(\sigma^*_i, \sigma^*_{-i})$ is an equilibrium we have: $u_i(\sigma^*_i,\sigma^*_{-i}) \ge u_i(\sigma_i,\sigma^*_{-i}) \quad \forall\, \sigma_i$, so $u_i(\sigma'_i, \sigma^*_{-i}) = u_i(\sigma^*_i, \sigma^*_{-i}) \ge u_i(\sigma_i, \sigma^*_{-i}) \quad \forall\, \sigma_i$. Similarly, $u_{-i}(\sigma'_i,\sigma'_{-i}) \ge u_{-i}(\sigma'_i,\sigma_{-i}) \quad \forall\, \sigma_{-i}$, so $u_{-i}(\sigma'_i,\sigma^*_{-i}) = u_{-i}(\sigma'_i,\sigma'_{-i}) \ge\ \ge u_{-i}(\sigma'_i,\sigma_{-i}) \quad \forall\, \sigma_{-i}$ (Note, that $u_{-i}(\sigma'_i, \sigma^*_{-i}) = u_{-i}(\sigma'_i, \sigma'_{-i})$ since $u_i(\sigma'_i,\sigma^*_{-i}) = u_i(\sigma'_i,\sigma'_{-i})$ and the game is zero-sum). This implies that $(\sigma'_i,\sigma^*_{-i})$ is a mixed strategy Nash equilibrium. Similarly it can be shown that $(\sigma^*_i, \sigma'_{-i})$ is a mixed strategy NE.

8.D.8 Let (σ_i,σ_{-i}) be a mixed strategy Nash equilibrium, and suppose in negation that σ_i assigns strictly positive probability to the pure strategies s^1_i and s^2_i, i.e. σ_i is not degenerate. This implies that s^1_i and s^2_i are each a best response to σ_{-i} and $u_i(s_i^1,\sigma_{-i}) = u_i(s_i^2, \sigma_{-i})$. By the convexity of S_i, $\alpha s_i^1 + (1-\alpha)s_i^2 \in S_i$, and since u_i is strictly quasiconvex we have that $u_i(\alpha s_i^1 + (1-\alpha)s_i^2, \sigma_{-i}) > u_i(s_i^1, \sigma_{-i}) = u_i(s_i^2,\sigma_{-i})$ for all $\alpha \in (0, 1)$. This contradicts the fact that s_i^1 and s_i^2 are each best response to σ_{-i}. Therefore, any mixed strategy NE of this game must be degenerate.

8.D.9 (a) Playing L or R is quite risky, since we do not know what player 1 will be playing. The risk of obtaining a payoff of -49 is very large compared to the payoff of 1 if player 2 played L, and the risk of obtaining a payoff of -100 is very large compared to the payoff of 2 if player 2 played LL or R. Therefore, it seems "reasonable" to play M.

(b) The two pure Nash equilibria of this game are (U,LL) and (D,R). To check for mixed strategy Nash equilibrium, player 1 must mix between U (with probability p) and D (with probability 1-p). Player 2 then has 11 possible mixing combinations: {LL,L}, {LL,M}, {LL,R}, {L,M}, {L,R}, {M,R}, {M,L,R}, {LL,M,R}, {LL,L,R}, {LL,L,M}, and {LL,L,M,R}. We will show that only the first combination, {LL,L}, is part of a mixed strategy NE.

For player 2 to mix between LL and L (with probabilities q and 1-q respectively), we must have that $p\cdot(2)+(1-p)\cdot(-100) = p\cdot(1)+(1-p)\cdot(-49)$ which yields $p = \frac{51}{52}$. The utility of player 2 from each strategy is then: $u_2(LL) = u_2(L) = \frac{1}{26}$, $u_2(M) = 0$, and $u_2(R) < 0$. Then, for player 1 to mix between U and D, we must have: $q\cdot(100)+(1-q)\cdot(-100) = q\cdot(-100)+(1-q)\cdot(100)$ which yields $q = \frac{1}{2}$, and $u_1(U) = u_1(D) = 0$. Therefore, $p = \frac{1}{26}$ and $q = \frac{1}{2}$ is a mixed strategy NE. For the rest of the answer we call this "NE*". We now show that no other mixing combination of player 2 can be part of a mixed strategy NE.

(i) If player 2 mixes with the combination {LL,M}, we must have $p = \frac{50}{51}$, which gives utilities $u_2(LL) = u_2(M) = 0$, and $u_2(L) = \frac{1}{51} > 0$, so this cannot be part of a mixed strategy NE.

(ii) If player 2 mixes with the combination {LL,R}, we must have $p = \frac{1}{2}$, which gives utilities $u_2(LL) = u_2(R) = -49$, and $u_2(M) = 0$, so this cannot be part of a mixed strategy NE.

(iii) If player 2 mixes with one of the combinations {L,M}, {L,R}, {M,R}, {M,L,R}, then player 1 will have D as a strict best response, which in turn has R as player 2's strict best response.

(iv) If player 2 mixes with one of the combinations {LL,M,R}, {LL,L,R}, or {LL,L,M,R}, then the analysis of **(ii)** above implies that this cannot be part of a mixed strategy NE.

(v) If player 2 mixes with the combination {LL,L,M} then the analysis of **(i)** above implies that this cannot be part of a mixed strategy NE.

(c) The choice in part **a)** is not part of any NE described above. It is easy to see that strategy M is rationalizable: If plyer 1 plays $p = \frac{1}{2}$ then M is the unique best response of player 2.

(d) If preplay communication is possible, the players can agree to play one of the pure strategy NE, which are payoff equivalent and Pareto dominant for both players. Therefore, player 2 will play either LL or R depending on the agreed upon equilibrium.

8.E.1 There are four pure strategies contingent on the type of player:

AA: Attack if either weak or strong type,

AN: Attack if strong and Not Attack if weak,

NA: Not Attack if strong and Attack if weak,

NN: Never attack.

The expected payoff of each pair of strategies can be easily computed and are given in Figure 8.E.1:

Player 1 \ Player 2	AA	AN	NA	NN
AA	$\frac{M}{4}-\frac{s+w}{2}, \frac{M}{4}-\frac{s+w}{2}$	$\frac{M}{2}-\frac{s+w}{4}, \frac{M}{4}-\frac{s}{2}$	$\frac{3M}{4}-\frac{s+w}{4}, \frac{-w}{2}$	$M, 0$
AN	$\frac{M}{4}-\frac{s}{2}, \frac{M}{2}-\frac{s+w}{4}$	$\frac{M-s}{4}, \frac{M-s}{4}$	$\frac{M}{2}-\frac{s}{4}, \frac{M-w}{4}$	$\frac{M}{2}, 0$
NA	$\frac{-w}{2}, \frac{3M}{4}-\frac{s+w}{4}$	$\frac{M-w}{4}, \frac{M}{2}-\frac{s}{4}$	$\frac{M-w}{4}, \frac{M-w}{4}$	$\frac{M}{2}, 0$
NN	$0, M$	$0, \frac{M}{2}$	$0, \frac{M}{2}$	$0, 0$

Figure 8.E.1

Any NE of this normal form game is a Baysian NE of the original game.

Case 1: M > w > s, and w > M/2 > s

From the above payoffs we can see that (AA,AN) and (AN,AA) are both pure strategy Bayesian Nash equilibria.

Case 2: M > w > s, and M/2 < s

From the above payoffs we can see that (AA,NN) and (NN,AA) are both pure strategy Bayesian Nash equilibria.

Case 3: w > M > s, and M/2 < s

From the above payoffs we can see that (AN,AN), (AA,NN) and (NN,AA) are pure strategy Bayesian Nash equilibria.

Case 4: w > M > s, and M/2 > s

From the above payoffs we can see that (AA,AN), (AN,AA) and (AN,AN) are pure strategy Bayesian Nash equilibria.

8.E.2 **(a)** Suppose that all the bidders, use the bidding function $b(v)$, that is if their valuation is v they bid $b(v)$. The expected payoff for a bidder whose valuation is v_i is given by:

$(v_i - b(v_i))\cdot Pr\{b(v_i) > b(v_j)\} + 0\cdot Pr\{b(v_i) < b(v_j)\}$. (Note that we ignore a tie since it is a zero probability event given that $b(v_i)$ is a monotonic linear function.) Since both players use the same monotonic linear bidding function then $Pr\{b(v_i) > b(v_j)\} = Pr\{v_i > v_j\} = v_i / \bar{v}$ (since the valuations are uniformly distributed on $[0,\bar{v}]$. $b(v)$ will in fact be the equilibrium bidding function if it is not better for a player to pretend that his valuation is different. To check this let us solve a bidders problem whose valuation is v_i and who has to decide whether he wants to pretend to have a different valuation v'. The bidder maximizes: $(v_i - b(v))\cdot(v / \bar{v})$, and the FOC is: $(v_i - b(v))/\bar{v} - b'(v)\, v/\bar{v} = 0$. $b(v)$ is an equilibrium bidding function if it is optimal for the bidder not to pretend to have a different valuation, that is, if $v = v_i$ is the optimal solution to the above FOC, i.e., if $(v_i - b(v_i))/\bar{v} - b'(v_i)\, v_i/\bar{v} = 0$. This is a differential equation that has to be satisfied by the bidding function $b(v)$ in order to be an equilibrium bidding function. The solution to this differential equation is $b(v) = v/2$. Thus a bidder whose valuation is v will bid v/2 (a monotonic linear function).

(b) We can proceed as above by assuming that all bidders use the same bidding function $b(v)$. Now, $Pr\{b(v_i) > b(v_j)\ \forall\ j \neq i\} = = Pr\{v_i > v_j\ \forall\ j \neq i\} = (v_i)^{I-1}/ \bar{v}$. Proceeding as in **a)** above, we get the following differential equation:

$(I-1)(v_i - b(v_i))(v_i)^{I-2}/\bar{v} - b'(v_i)\,(v_i)^{I-1}/\bar{v} = 0$. The solution to this differential equation is $b(v) = \frac{I-1}{I}\cdot v$. As $I \to \infty$, $b(v) = \frac{I-1}{I}\cdot v \to v$, i.e., as the number of players goes to infinity each player will bid his valuation.

8.E.3 A firm of type i = H or L, will maximize its expected profit, taken as given that the other firm will supply q_H or q_L depending whether this firm is of type H or L. A type $i \in \langle H,L \rangle$ firm 1 will maximize:

$$\max_{q_i^1} (1-\mu)[(a - b(q_i^1 + q_H^2) - c_i)q_i^1] + \mu[(a - b(q_i^1 + q_L^2) - c_i)q_i^1]$$

The FOC yields: $(1-\mu)[a - b(2q_i^1 + q_H^2) - c_i] + \mu[a - b(2q_i^1 + q_L^2) - c_i] = 0$. In a symmetric Bayesian Nash equilibrium: $q_H^1 = q_H^2 = q_H$ and $q_L^1 = q_L^2 = q_L$. Plugging this into the F.O.C we get the following two eqations:

$$(1-\mu)[a - 3b\, q_H - c_H] + \mu[a - b(2q_H + q_L) - c_H] = 0,$$

$$(1-\mu)[a - b(q_H + 2q_L) - c_L] + \mu[a - 3b\, q_L - c_L] = 0.$$

Therefore, we obtain that

$$q_H = \left[a - c_H + \frac{\mu}{2}(c_L - c_H)\right]\cdot\frac{1}{3b},$$

$$q_L = \left[a - c_L + \frac{1-\mu}{2}(c_H - c_L)\right]\cdot\frac{1}{3b}.$$

8.F.1 For the proof of Proposition 8.F.1, we refer to:

Selten, R. (1975) "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," *International Journal of Game Theory.*

Another good source is Section 8.4 in Fudenberg & Tirole, (1991) *Game Theory*, MIT press.

8.F.2 For the solution of this question we refer to:

van Damme, E. (1983). *Refinements of the Nash Equilibrium Concept.* Berlin: Springer-Verlag (pp. 28-31).

8.F.3 For the proof of this statement, we refer to:

Selten, R. (1975) "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," *International Journal of Game Theory*.

Another good source is Section 8.4 in Fudenberg & Tirole, (1991) *Game Theory*, MIT press.

CHAPTER 9

9.B.1 There are 5 subgames, each one beginning at a different node of the game (this includes the whole game itself).

9.B.2 **(a)** Clearly if σ° is a Nash equilibrium of Γ_E, and Γ_E is the only proper subgame of Γ_E, then σ° induces a NE in every proper subgame of the game Γ_E. Thus, by definition σ° is a subgame perfect NE of Γ_E.

(b) Assume in negation that σ° is a subgame perfect equilibrium of Γ_E, but it does not induce a subgame perfect Nash equilibrium in every proper subgame of Γ_E. Then there exists a proper subgame (say, Π_E) of Γ_E in which the restriction of σ° to Π_E is not a SPNE. This implies that there exists a proper subgame (say, Ω_E) of Π_E in which the restriction of σ° to Ω_E is not a NE. Since Ω_E is a proper subgame of Π_E and Π_E is a proper subgame of Γ_E, then Ω_E is also a proper subgame of Γ_E. Therefore, σ° does not induce a NE in a proper subgame of Γ_E - contradiction.

9.B.3 Let player 1's pure strategy be $s_1 \in \{L,R\}$, player 2's be $s_2 \in \{a,b\}$, and player 3's be $s_3 = (x,y,z)$ where $x,y,z \in \{l,r\}$, x is what 3 does after player 1 played L, y is what 3 does after 1 played R and 2 played a, and z is what 3 does after 1 played R and 2 played b. The pure strategy SPNE identified in the example is (R,a,(r,r,l)), which is easily seen to be a NE. Three other NE which are not SPNE but yield the same outcome are (R,a,(l,r,l)), (R,a,(l,r,r)) and (R,a,(r,r,r)). For each of these NE, player 3 is not choosing a rational move for some of his nodes. If player 3 is reached, he will always do the best thing for himself, therefore if 1 plays L, 3 will play r. To support this as a

NE we need strategies for 2 and 3 that give player 1 less than -1 in the subgame starting from player 2's node, but recall that these strategies need not be subgame perfect. Therefore, (L,b,(r,r,r)) would be another NE, in which player 3 is again not acting sequentially rational. There can be no NE with player 2 being reached and then choosing b, since after this move 3 will choose l, giving 2 a payoff of -1. Player 2 will then prefer to deviate and play a, resulting in a higher payoff no matter what 3 will do then.

9.B.4 Proposition 9.B.1 claims that in any game, in which no player has the same payoffs at any two terminal nodes, there exists a unique SPNE. Now suppose one of the players has the same payoff at one of the terminal nodes. This means that he will be indifferent between two actions that lead to either one of them. However, since +he game is zero-sum, the other player will also be indifferent between the payoffs resulting from these two actions. Thus, in a finite zero-sum game of perfect information, there may exists many different (in terms of the strategies played) SPNE (because of potential indifference of the player between different terminal nodes), but all of them will yield the same payoffs for the players, i.e. there are unique SPNE payoffs.

9.B.5 Note: for parts **(a)** and **(b)**, m_i denotes the number of strategies that player i has. For the remainder of the question, m_i denotes the move of player i. This is done to be consistent with the question.

(a) Since the game is simultaneous, each player i has m_i pure strategies. If we allow for mixed strategies then each player has a continuum of strategies: $\Sigma_i = \{ (p_1,\dots p_{m_i}) \in \mathbb{R}_+^{m_i} : p_k \geq 0 \ \forall\ k,\ \sum_{k=1}^{m_i} p_k = 1 \}$.

(b) Since player 1 moves first, he cannot make his strategies contingent on any history, thus he still has m_1 (pure) strategies. Player 2 can, however, condition her play on player 1's move, thus allowing her to specify one of m_2 actions for each of player 1's m_1 plays. Therefore, she has $(m_2)^{m_1}$ (pure) strategies. There is, of course, a continuum of mixed strategies.

(c) Assume in negation that for all (m_1,m_2) and (m_1',m_2') where either $m_1 \neq m_1'$ or $m_2 \neq m_2'$ we have that $\phi_i(m_1,m_2) \neq \phi_i(m_1',m_2')$ for both i=1,2. Then, due to lack of indifference (the negation condition) player 2 will have a unique best response at each of her nodes. By backward induction, and by lack of indifference, player 1 will have a unique best response to the SPNE strategy of player 2, which contradicts multiple SPNE.

(d) Since player 2 has no indifference, she will have a unique best response after she is reached. Let (m_1^*,m_2^*) be the NE for the game in **(a)** yielding a payoff of π_1 to player 1. Clearly, player 1 playing m_1^* and player 2 playing m_2^* at each of her nodes is a NE in the extensive form game. However, this is not necessarily a SPNE since player 2 is not necessarily playing a best response at each of her nodes. Note that in any SPNE player 2 will play m_2^* after player 1 plays m_1^*, therefore player 1 can promise himself a payoff of π_1. Given player 2's unique SPNE strategy in the sequential game, player 1 can therefore do at least as well as the NE (m_1^*,m_2^*) in any SPNE. This conclusion would not necessarily hold for NE of the sequential game - low payoffs for player 1 could be sustained in a NE with incredible threats by player 2.

(e) (1) Consider the normal and extensive form versions of a game depicted in Figure 9.B.5(e.1):

		Player 2 L	Player 2 R
Player 1	U	0,1	1,1
	D	0,0	1,0

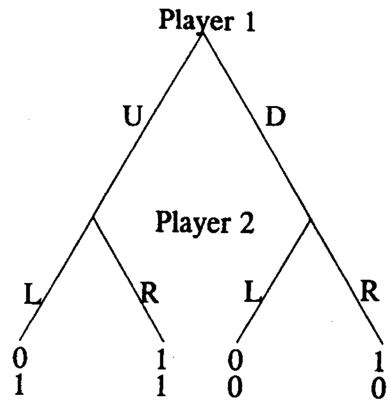


Figure 9.B.5(e.1)

In this game condition (ii) holds for some strategy pairs. It is easy to see (by backward induction) that any path in the extensive form game can be supported by a SPNE. Therefore, if we consider the NE (U,R) in the normal form version, then the conclusion of part **(d)** does not hold.

(2) Consider the normal and extensive form versions of the "matching pennies" game depicted in figure 9.B.5(e.2):

		Player 2 L	Player 2 R
Player 1	U	1,0	0,1
	D	0,1	1,0

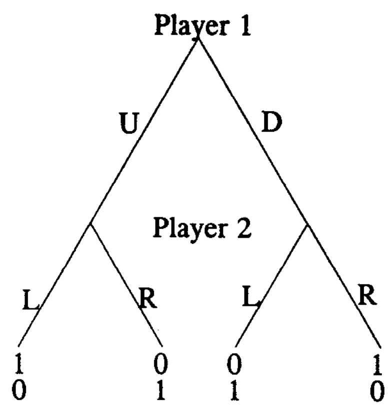


Figure 9.B.5(e.2)

The unique mixed strategy NE in the normal for game gives each player an expected payoff of 1/2. It is easily seen, however, that the only two SPNE in the extensive form game, (U,(R,L)) and (L,(R,L)), give player 1 a payoff of 0. Again, the conclusion of part **(d)** does not hold.

9.B.6 To find the mixed strategy equilibrium of the post-entry subgame, suppose that firm E plays Small with probability x and Large with probability 1-x; firm I plays Small with probability y and Large with probability 1-y. For firm I to be indifferent between playing Small and Large we need: $-6x - 1(1-x) = 1x - 3(1-x)$, or $x = 2/9$. For firm E to be indifferent between playing Small and Large we need: $-6y - 1(1-y) = 1y - 3(1-y)$, or $y = 2/9$. Thus in the mixed strategy equilibrium of the post-entry game firms E and I play Small with probability 2/9. This gives both firms a payoff of -19/9, which will cause firm E to choose not to enter. Therefore, the following strategies constitute a SPNE: firm E plays Out at the first node and randomizes between Small and Large in the second node (with probabilities 2/9 and 7/9 respectively). Firm I plays Small with probability 2/9 and Large with probability 7/9 given that firm E entered.

9.B.7 Consider the last period of the game. The offering player will offer (v, 0) and the offered player will accept. In the second to last period, the offering player will offer the other player a share that will make him indifferent between accepting now and rejecting and obtaining v in the next period. With the given cost c for making an offer, the offered player in the second to last period will accept any offer that gives him at least v - c. Thus, the offering player in the second to last period offers (c, v-c). Similarly, we can show that the offering player in the third to last period offers (v, 0).

If T is odd, then the player 1 will be the last player to make an offer. Thus, he will offer (v, 0) in every period in which he makes offers and accept an offer if he obtains at least v-c. Player 2 will offer (c, v-c) in every period in which she makes offers and accept if she obtains a non-negative payoff. The result is that in the first period player 1 will offer (v, 0) and player 2 will accept.

If T is even, then the player 2 will be the last player to make an offer. Thus she will offer (0, v) in every period in which she makes offers and accept an offer if she obtains at least v-c. Player 1 will offer (c, v-c) in every period in which he makes offers and accept if he obtains a non-negative payoff. The result is that in the first period player 1 will offer (c, v-c) and player 2 will accept.

The argument above holds for any finite game, but if T = ∞, then there can be many SPNE of this game. For an analysis of this case we refer to: Rubinstein, A. (1982) "Perfect Equilibria in a Bargaining Model," *Econometrica*, 50:97:109.

9.B.8 Take all the proper subgames that do not strictly contain another proper subgame. Since the whole game is finite, all proper subgames must be finite. Thus, by Proposition 8.D.2, there exists a mixed strategy NE in each of these subgames. Now construct a new finite game with terminal nodes at the root of each of the previous subgames, and associate the payoff for every player for each such terminal node by the payoff obtained from playing one of the mixed strategy NE in the subgame that followed in the original game. We can again look for all the proper subgames that do not strictly contain another proper subgame. Repeating the above process we can find the mixed strategy Nash equilibria in these subgames. Since the game is finite, repeating the above procedure will end after a finite number of rounds. We have therefore constructed a SPNE of the game. In every subgame players will play the strategies that constitute one of the mixed strategy Nash equilibria in this subgame.

9.B.9 The pure strategy NE of the one-shot game are (a_2,b_2) and (a_3,b_3). Thus any SPNE involves playing either of these in the second period. Thus, playing either of these strategies in both periods constitutes a SPNE. Additionally, the players could use them in any combination in the two periods. This results in the following two classes of SPNE (a total of four SPNE):

1) Player 1 plays a_i and player 2 plays b_i in both periods, $i \in \{2,3\}$.

2) Player 1 plays a_i in the first period and a_j in the second period; player 2 plays b_i in the first period and b_j in the second period, $i,j \in \{1,2\}$ and $i \neq j$.

However, there exist more SPNE in this game. The reason is that player 1 (or 2) can punish the other player by playing a_3 (b_3) in the second period, if the other player did not cooperate in the first period. [Note that this can only happen because there are more than one NE in the second stage]. This gives rise to two more classes of SPNE:

3) Player 1's strategy: Play a_i, $i \in \{1,2,3\}$ in period 1; Play a_2 in period 2 if player 2 played b_1 in period 1, otherwise play a_3.
Player 2's strategy: Play b_1 in period 1. Play b_2 in period 2 if player 1 played a_i in period 1, otherwise play b_3.

4) Player 2's strategy: Play b_i, $i \in \{1,2,3\}$ in period 1; Play b_2 in period 2 if player 1 played a_1 in period 1, otherwise play b_3.
Player 1's strategy: Play a_1 in period 1. Play a_2 in period 2 if player 2 played b_i in period 1, otherwise play a_3.

Too check that each of the 6 SPNE described by these classes in indeed a SPNE, note that by deviating a player looses 4 in the second period (no

CHAPTER 13

13.B.1 The three functions are graphed in figure 13.B.1(a). This graph also depicts the situation in figure 13.B.1 in the text.

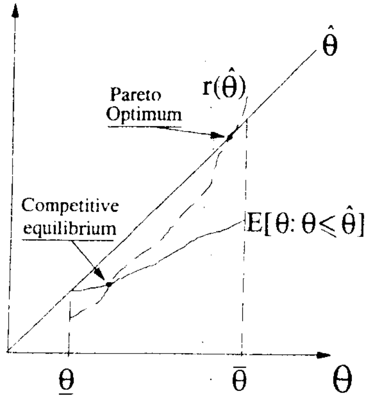


Figure 13.B.1(a)

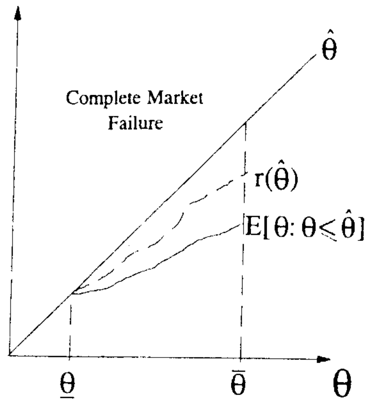


Figure 13.B.1(b)

Figure 13.B.1(b) depicts the situation in figure 13.B.2 in the text.

Figure 13.B.1(c) depicts the situation in figure 13.B.3 in the text.

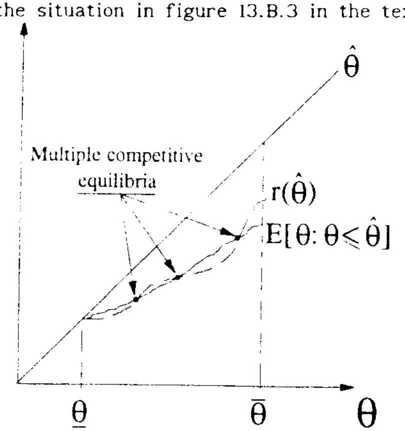


Figure 13.B.1(c)

13.B.2 If $w = \hat{\theta}$, only workers of type $\theta \le \hat{\theta}$ will accept the wage w and work. But $E[\theta \mid \theta \le \hat{\theta}] < \hat{\theta} = w$, which implies that firms demand less worker than there are in supply. This implies that the market does not clear. If $w > \hat{\theta}$,

only workers of type $\theta \le \theta^* > \hat{\theta}$, with $r(\theta^*) = w$, will accept the wage w and work (since $r()$ is an increasing function). But $E[\theta \mid \theta \le \theta^*] < \theta^* = w$, which implies that firms demand less worker than there are in supply. This implies that the market does not clear.

Thus, to obtain market clearing firms have to offer a wage $w < \hat{\theta}$, which implies that some workers of type $\theta < \hat{\theta}$ will not work, and there will be underemployment in the competitive equilibrium (in an equilibrium with perfect information all workers of type $\theta \le \hat{\theta}$ will work).

13.B.3 **(a)** Suppose firms offer a wage of w. All workers of type θ, with $r(\theta) \le w$, will accept the wage and work. Suppose there exists a θ^* with $r(\theta^*) = w$. Then all workers of type $\theta \ge \theta^*$ will work, since $r(\theta) \le r(\theta^*) = w$ and $r()$ is decreasing. Thus, the more capable workers are the ones who will work at any given work.

(b) Firms can offer the wage $w = \bar{\theta}$, and since $r(\bar{\theta}) > \bar{\theta}$ no workers of type $\bar{\theta}$ will work. From part **(a)**, no worker of any type will work. Therefore, the competitive equilibrium is Pareto efficient, i.e. nobody will work.

(c) If $w = \hat{\theta}$, only workers of type $\theta \ge \hat{\theta}$ will accept the wage w and work. But $E[\theta \mid \theta \ge \hat{\theta}] > \hat{\theta} = w$, which implies that firms demand more workers than there are in supply, and the market will not clear. If $w < \hat{\theta}$, only workers of type $\theta \ge \theta^* > \hat{\theta}$, with $r(\theta^*) = w$, will accept the wage w and work (since $r()$ is a decreasing function). But $E[\theta \mid \theta \ge \theta^*] > \theta^* = w$, which implies that firms demand more workers than there are in supply, and the market will not clear.

Thus, to obtain market clearing, firms have to offer a wage $w > \hat{\theta}$, which implies that some workers of type $\theta < \hat{\theta}$ will accept the job, and there is over employment in the competitive equilibrium (in an equilibrium with perfect information only workers of type $\theta \ge \hat{\theta}$ will work).

13.B.4 We can think of the true valuation of the good, y, as a state of nature $s \in S$ where S is the set of all states of nature, and both agents 1 (say the seller) and 2 (say the buyer) have a common prior that is common knowledge. Let $H_i(s)$ be the set of states that agent i believes is possible given the true state s (this will depend on the signal observed). We can define the following two events:

$$T_1 \equiv \left\{ s \in S \;\middle|\; E[\, y \mid H_1(s) \cap T_2 \,] \leq p \right\}$$

$$T_2 \equiv \left\{ s \in S \;\middle|\; E[\, y \mid H_2(s) \cap T_1 \,] \geq p \right\}$$

That is, T_i is the event that agent i will say "trade" given that he knows that event $H_i(s)$ has occurred, and that he believes agent j will say "trade". Assume that there exists an equilibrium where both agents say "trade". Then, each agent i *knows* that event T_i occurred, and since this is an equilibrium then each agent i *believes* with probability 1 that agent j knows that event T_j occurred. Therefore, each agent believes with probability 1 that event $T = T_1 \cap T_2$ occurred. The seller (1) then prefers to trade if and only if $E[y|T] \leq p$, and the buyer (2) prefers to trade if and only if $E[y|T] \geq p$. Assuming a generic distribution of values both can hold with probability zero, therefore the set T must occur with probability zero.

13.B.5 (a) When $r(\theta)=r$ for all θ, and $E[\theta] \geq r > \underline{\theta}$, then the firms will make zero profits (a necessary condition for a competitive equilibrium) in two cases: Either $w^*=E[\theta]$ or $w^*=\underline{\theta}$. In this case, if $w^*>E[\theta]$ then firms are losing money, while if $w^* \in [r,E(\theta))$ then $\Theta^*=[\underline{\theta},\bar{\theta}]$ and firms make positive profits. Finally if $w^* \in (\underline{\theta},r)$ then no one will accept employment, and together with the assumption that in this case firms believe that any worker who might accept employment is a $\underline{\theta}$ type worker, we must have the offered wage to be $\underline{\theta}$.

If, however, $\underline{\theta} \geq r$, then when $w^*=\underline{\theta}$ firms will make positive profits since all workers will work, and $E(\theta)>w^*$, so the only competitive equilibrium is $w^*=E(\theta)$. On the other hand, if $r>E(\theta)$ then when $w^*=E(\theta)$ no worker will wish to be employed, and due to the assumption on firm beliefs, we must have $w^*=\underline{\theta}$.

(b) In the equilibrium with $w^*=E(\theta)$, both firms have zero profit, there is full employment, and all workers have a utility of $E(\theta)>r$. If, however, $w^*=\underline{\theta}$ then again both firms have zero profits but workers have a utility of only r, this is Pareto dominated by the full employment equilibrium.

(c) The analysis of SPNE is as in proposition 13.B.1 which applies (with straightforward modifications due to a constant r() function). When $E(\theta)=r$, then both competitive equilibria are SPNE. This follows because no firm can offer a wage $w \in [\underline{\theta},r]$ and make positive profits (because if $w<E(\theta)=r$ no worker will accept employment) and a wage $w>r$ will cause losses.

(d) Clearly, when $E(\theta) \geq r > \underline{\theta}$ and when $\underline{\theta} \geq r$ then the highest wage competitive equilibrium has full employment, and is therefore Pareto optimal. For the case where $r>E(\theta)$ a simpler version of proposition 13.B.2 can be applied.

13.B.6 For a similar analysis we refer to:

Wilson, C. (1980) "The Nature of Equilibrium in Markets with Adverse Selection," *The Bell Journal of Economics*, 11:108-30.

13.B.7 First assume there is a Pareto improving market intervention $(\tilde{w}_e,\tilde{w}_u)$ that reduces employment with respect to a competitive equilibrium with wage w^*. Clearly, we cannot have $\tilde{w}_e < w^*$ since then those workers who are employed are worse off. Similarly we cannot have $\tilde{w}_u < 0$. Now assume that $\tilde{w}_e > w^*$ and $\tilde{w}_u > 0$. We can then reduce both $\tilde{w}_e$ and $\tilde{w}_u$ by $\varepsilon > 0$ such that $\tilde{w}_e - \varepsilon > w^*$ and $\tilde{w}_u - \varepsilon > 0$ will still hold, the same groups of agents will be employed/unemployed, and the government now has a surplus (it broke even with $(\tilde{w}_e,\tilde{w}_u)$ since it must have a balanced budget). The government can then distribute this surplus by raising only $\tilde{w}_u$, and this would still be a Pareto improvement relative to the competitive equilibrium. This can be done as long as $\tilde{w}_e > w^*$, and since it was employment reducing we must have $\tilde{w}_e - w^* < \tilde{w}_u - 0$, implying that we can reduce both $\tilde{w}_e$ and $\tilde{w}_u$ until $\tilde{w}_e = w^*$. The result is a Pareto improving intervention of the form $(\hat{w}_e,\hat{w}_u)$ with $\hat{w}_e = w^*$ and $\hat{w}_u > 0$. The converse is trivial: if $(\hat{w}_e,\hat{w}_u)$ is a Pareto improving market intervention of the form $\hat{w}_e = w^*$ and $\hat{w}_u > 0$, then we must have less employment since the marginal type θ^* who decided to work had $w^* = r(\theta^*)$ and now can be unemployed and receive $r(\theta^*) + \hat{w}_u > w^*$ so he will decide to be unemployed, and by continuity a positive mass of types will decide so as well.

Second, assume there is a Pareto improving market intervention $(\tilde{w}_e,\tilde{w}_u)$ that increases employment with respect to a competitive equilibrium with wage w^*. Clearly, we cannot have $\tilde{w}_e < w^*$ since then those workers who are employed are worse off. Similarly we cannot have $\tilde{w}_u < 0$. Now assume that $\tilde{w}_e - \varepsilon > w^*$ and $\tilde{w}_u - \varepsilon > 0$. We can then reduce both $\tilde{w}_e$ and $\tilde{w}_u$ by $\varepsilon > 0$ such that $\tilde{w}_e > w^*$ and $\tilde{w}_u > 0$ will still hold, the same groups of agents will be employed/unemployed, and as before, the government can distribute the generated surplus by raising only $\tilde{w}_e$, and this would still be a Pareto improvement relative to the competitive equilibrium. This can be done as long as $\tilde{w}_u > 0$, and since it was employment increasing we must have $\tilde{w}_e - w^* > \tilde{w}_u - 0$, implying that we can reduce both $\tilde{w}_e$ and $\tilde{w}_u$ until $\tilde{w}_u = 0$. The result is a Pareto improving intervention of the form $(\hat{w}_e,\hat{w}_u)$ with $\hat{w}_e > w^*$ and $\hat{w}_u = 0$. As before, the converse is trivial.

Finally, these facts give a simple proof of proposition 13.B.2. by contradiction: If there were a Pareto improving intervention, it must be either employment increasing or reducing. Therefore, it must be of one of the two forms: (i) $(\hat{w}_e,\hat{w}_u) = (w^* + \delta, 0)$, or (ii) $(\hat{w}_e,\hat{w}_u) = (w^*, \varepsilon)$. If it is of

form (i), then a firm could have deviated, proposed $w^* + \frac{\delta}{2}$, and by the conditions given would have made positive profits - a contradiction to w^* being a SPNE. If it is of form (ii) then the government cannot balance its budget since $r(\cdot)$ is strictly increasing, therefore the best of the formerly employed types will choose to be unemployed, and by paying w^* to the remaining employed workers the government must lose money, in addition to the unemployment benefit it is paying out, a contradiction to (w^*, ε) being a feasible intervention.

13.B.8 Assume that we have a model similar to the one studied in section 13.B as displayed in figure 13.B.8 (let $r(\theta^*) = w^*$).

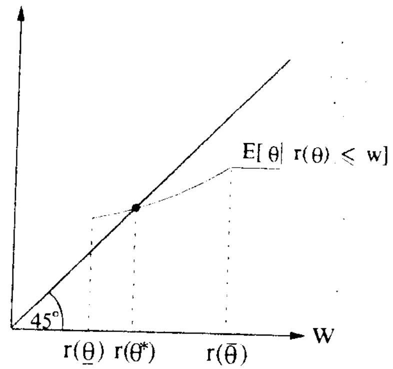


Figure 13.B.8

In our new model, since all workers of type $\theta \in [\theta^*, \theta]$ don't work, they get to consume the product x. Since the government observes the level of consumption of x, and prior to intervention the economy is in a competitive equilibrium, the government is able to identify the type of all individuals for which $\theta > \theta^*$. (Because their consumption is a function of θ, $x(\theta)$, which is known to the government, and $x(\theta)$ is invertible because it is increasing.) In the competitive equilibrium of the model of section 13.B., firms break even, and workers' payoffs are:

is still an increasing function. Therefore, the details of the proof in the textbook go through.

13.C.1 Suppose some workers, whose types do not equal to $\underline{\theta}$, do not submit to the test. Call the worker with the highest type that did not submit to the test $\theta^* > \underline{\theta}$ Firms will now offer all the workers that did not submit to the test a wage justified for the mean worker that did not submit to the test, i.e. they will offer a wage equal to the expected value of θ for workers that did not submit to the test. This expected value will be lower than the type of worker θ^*, while if worker θ^* submits to the test he will receive a wage of θ^* which is higher than the wage he would receive if he would not submit to the test. Therefore all workers (except for the lowest type worker) submit to the test in the unique SPNE of this game. It follows that firms offer no more than $\underline{\theta}$ to workers who do not submit to the test (they loose money otherwise).

13.C.2 For simplicity, assume μ=1 (this does not change the qualitative results). The competitive equilibrium with perfect information is given in the following Claim:

Claim 1: At the competitive equilibrium with perfect information, type θ_H (resp. θ_L) gets education level e_H^* (resp. e_L^*), where $\frac{dc}{de}(e_H^*, \theta_H) = \theta_H$. (resp. $\frac{dc}{de}(e_L^*, \theta_L) = \theta_L$). The wage for θ_H (resp. θ_L) is given by $w_H^* = \theta_H + \theta_H e_H^*$ (resp. $w_L^* = \theta_L + \theta_L e_L^*$).

Proof of Claim 1: If a worker of quality θ gets education level e, then his marginal productivity is $\theta + \theta e$, and his wage will be equal to $\theta + \theta e$, since firms are competitive. Workers of type θ will thus choose their level of education to maximize their utility, given this wage level:

$$\max_e \; w - c(e, \theta) = \theta + \theta e - c(e, \theta)$$

and the FOC is: $\frac{dc}{de}(e^*, \theta) = \theta$. Q.E.D.

Remark: The competitive equilibrium with perfect information is Pareto efficient.

As you may expect, both the separating equilibrium and the pooling equilibrium look similar as in the original model in which education does not affect productivity. That is:

- the equilibrium contracts provide the firms with zero profits.
- the low productivity type will obtain the optimal level of education in a separating equilibrium.

The good news is that the high productivity type may also obtain the optimal level of education, i.e the above competitive equilibrium with perfect information may emerge as a separating equilibrium. In Figure 13.C.2(a) the competitive equilibrium with perfect information is sustained as a separating equilibrium:

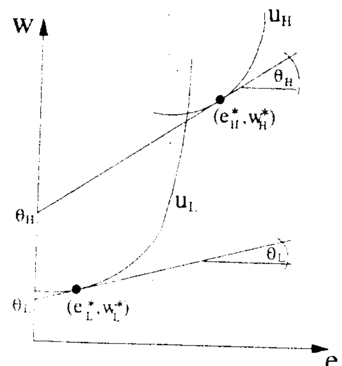


Figure 13.C.2(a)

In Figure 13.C.2(b) below, the outcome of the competitive equilibrium cannot be attained as an equilibrium with imperfect information. Type θ_L would like to pretend to be of type θ_H, if he is offered $(e_L^*, \theta_L + \theta_L e_L^*)$ and $(e_H^*, \theta_H + \theta_H e_H^*)$. Thus, the incentive constraint for the low productivity type is not satisfied.

the utilities of both types are $u_L(w_L,e_L) = 0.5$, and $u_H(w_H,e_H) = 3.47$. Note that since $E(\theta) = 3\frac{1}{3} < 3.45 = u_H(4,1)$, we would not get a Pareto improvement by banning the signal since the H type would be worse off.

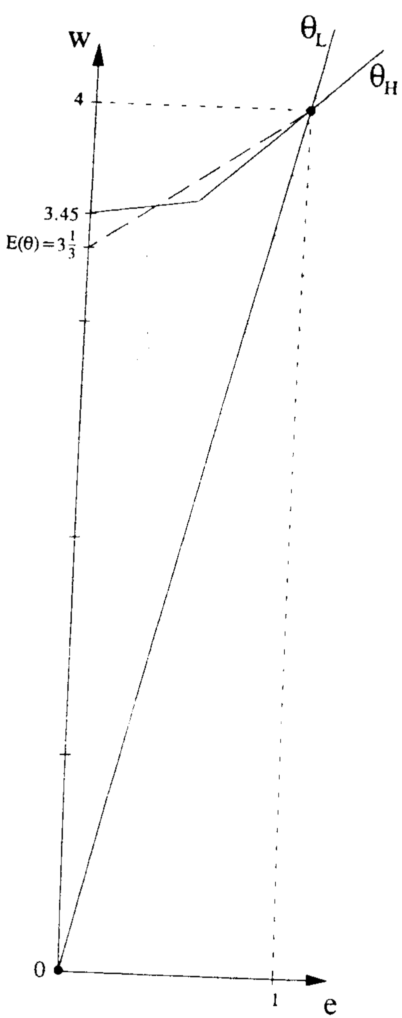


Figure 13.C.3(a)

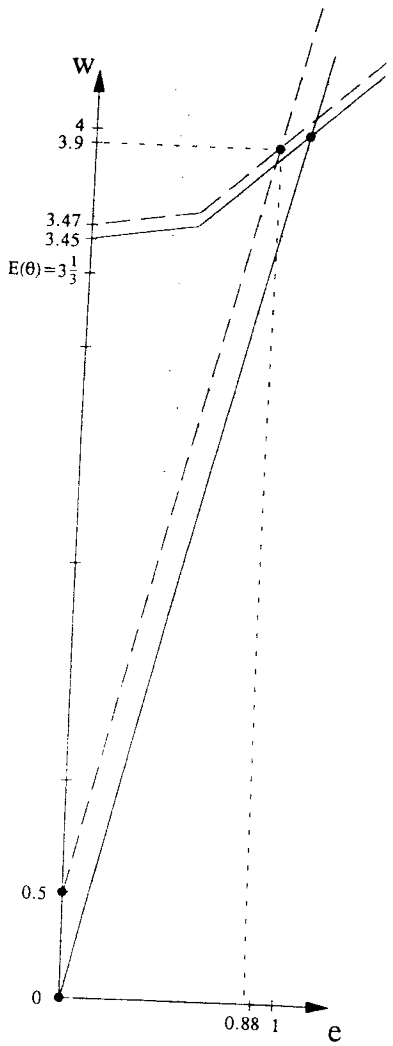


Figure 13.C.3(b)

13.C.4 [First Printing Errata: the question should end with "Derive the (nuique) separating perfect Bayesian equilibrium."]

The firms will offer a wage $w(e)$ as a function of the observed level of education. Given a wage schedule $w(\cdot)$, an agent of type θ^* chooses e to maximize his utility $w(e) - e^2/\theta^*$, and the FOC is: $w'(e^*) = \frac{2e^*}{\theta^*}$. Since the firms are competitive we must have zero profits, $w(e^*) = \theta^*$, in equilibrium. Combining this with the FOC gives us the differential equation: $w(e^*)\cdot w'(e^*) = 2e^*$, which implies that $w(e) = \sqrt{2}\cdot e$. This wage function, together with each type choosing e that satisfies the FOC above, that is, $e(\theta) = \theta / \sqrt{2}$, is the unique separating PBE.

There also exist many (in fact, a continuum of) pooling equilibria (w^*,e^*) of the form: $w^* = E(\theta)$, and $e \in [0,\hat{e}]$, where $\hat{e}$ is calculated by equating: $u(w^*,\hat{e}|\underline{\theta}) = u(\underline{\theta},0|\underline{\theta})$.

13.C.5 **(a)** The consumer will buy the product if the expected value of the product is higher than the price, i.e., if $\lambda v_H + (1-\lambda) v_L \geq p$.

(b) Suppose there exists a separating equilibrium in which the high quality producers spends A on advertising and only the high quality product will be bought (in a separating equilibrium consumers know the quality of a product, so low quality products will not be bought since $p > v_L$). This implies that the low quality producer makes no profit and the high quality producer makes a non-negative profit, $\Pi_H = p - c_H - A \geq 0$.

However, a low quality producer can make a positive profit buy spending A on advertising, since the consumer will then mistake him for a high quality producer and buy the good from him. The low quality producer's profit will equal $\Pi_L = p - c_L - A > p - c_H - A \geq 0$. Therefore, no separating equilibrium can exist.

(Note, that the banks will not be compensated for the risk that they assume since they are risk-neutral): $\lambda(p_G R + (1-p_G)0) + (1-\lambda)(p_B R + (1-p_B)0) = 1 + r$, or, $R = (1+r) / (\lambda p_G + (1-\lambda)p_B)$.

An entrepreneur of type i will pursue a project if: $(p_i(\Pi-R) + (1-p_i)0) \geq 0$ for $i \in \{G,B\}$, or if $\Pi \geq (1+r)/[\lambda p_G + (1-\lambda)p_B]$. Since, by assumption $\Pi \geq (1+r)/p_G$ and $\Pi \leq (1+r)/p_B$, the entrepreneurs will pursue a project if λ is large enough. In other words, if the fraction of good projects (λ) is large enough, then the banks will set a lower interest rate (R) and thus, the entrepreneurs will undertake the project, i.e. $\Pi \geq R = (1+r)/[\lambda p_G + (1-\lambda)p_B]$.

(b) **(i)** The entrepreneur's expected payoff from a project of type $i \in \{G,B\}$ is: $p_i[\Pi-(1-x)R] + (1-p_i)0 - (1+\rho)x = p_i(\Pi-R) - x[(1+\rho) - p_i R]$.

(ii) In a separating equilibrium the banks will know the type of the project. Thus, the banks will offer an entrepreneur that pursues a good project an interest rate of $R = (1+r)/p_G \leq \Pi$ and an entrepreneur that pursues a bad project an interest rate of $R = (1+r)/p_B \geq \Pi$. Therefore, in a separating equilibrium no bad projects will be pursued.

Therefore, the minimum level of x, that allows the entrepreneur with the good project to signal his type, has to give an entrepreneur with a bad project a negative expected payoff if he contributes this level x of internal funds and obtains financing from the bank at an interest rate of $R = (1+r)/p_G$. That is, $p_B(\Pi-R) - x((1+\rho) - p_B R) = 0$. Substituting in the equilibrium level of R we get: $p_B[\Pi-(1+r)/p_G] - x[(1+\rho) - p_B(1+r)/p_G] = 0$, or, $x = [(\Pi-(1+r)/p_G)]/[(1+\rho)/p_B - (1+r)/p_G] < 1$.

It follows that as p_G increases and r decreases, x increases. As Π or p_B increase, x increases. Since an entrepreneur with a bad project will obtain a zero expected payoff by pursuing the project, an entrepreneur with a good project will obtain a positive payoff.

The separating equilibrium is then: Entrepreneurs with bad projects will

contribute $x = 0$ and accept a bank's offer if $R \leq \Pi$. Entrepreneurs with good projects will contribute $x = [(\Pi-(1+r)/p_G)]/[(1+\rho)/p_B - (1+r)/p_G]$ and accept a bank's offer if $R \leq (1+r)/p_G$. The banks will offer an interest rate of $R = (1+r)/p_B$ if the entrepreneur contribute $x = 0$ and $R = (1+r)/p_G$ if he contributed $x = [(\Pi-(1+r)/p_G)]/[(1+\rho)/p_B - (1+r)/p_G]$.

(iii) The entrepreneurs with the bad project will not be better off, and may be worse off in the separating equilibrium of part **(ii)**. For large enough λ, all projects will be financed in the equilibrium of part **(i)** and the entrepreneurs with the bad project will make a strictly positive expected profit. In the separating equilibrium of part **(ii)**, however, entrepreneurs with bad projects will always obtain a payoff of zero. For small λ, entrepreneurs with bad projects will obtain a payoff of zero in both equilibria.

For small λ, entrepreneurs with good projects will be better off in the separating equilibrium, since they obtain a positive payoff in the separating equilibrium and a zero payoff in the equilibrium of part **(i)** (since the projects will not be financed).

As λ becomes larger, projects will also be funded in the equilibrium of part **(i)**. Now, the entrepreneur will have to pay a higher interest rate R on the bank loan in the equilibrium of part **(i)**. In the separating equilibrium, however, the entrepreneur has to contribute his own funds, which is costly to him since he is liquidity constraint. Thus as λ becomes large enough, the entrepreneur will be better of in the equilibrium of part **(i)**.

13.D.1 This is the screening analog of the signaling model of Exercise 13.C.2. The competitive equilibrium with perfect information is given in the following Claim:

<u>Claim 1</u>: At the competitive equilibrium with perfect information, type θ_H

(resp. θ_L) has a task level of t_H^* (resp. t_L^*), where $\frac{dc}{dt}(t_H^*, \theta_H) = \theta_H$. (resp. $\frac{dc}{dt}(t_L^*, \theta_L) = \theta_L$. The wage for θ_H (resp. θ_L) is given by $w_H^* = \theta_H + \theta_H t_H^*$ (resp. $w_L^* = \theta_L + \theta_L t_L^*$) .

Proof of Claim 1: A worker of quality θ performing a task level of t, produces $\theta + \theta t$, and his wage will be equal to $\theta + \theta t$, since firms are competitive. Workers of type θ will thus choose their task level to maximize their utility, given this wage level: max $w - c(t, \theta) = \theta + \theta t - c(t, \theta)$, and the FOC is: $\frac{dc}{dt}(t^*, \theta) = \theta$. Q.E.D.

Remark: The competitive equilibrium with perfect information is Pareto efficient.

As one may expect, the equilibrium looks similar as in the original model in which the task level is unproductive. That is:

- the equilibrium contracts provide the firms with zero profits.
- there exists no pooling equilibrium
- the low productivity type will provide the optimal task level in a separating equilibrium.

The good news is that the high productivity type may also obtain the optimal task level, i.e the above competitive equilibrium with perfect information may emerge as a separating equilibrium. In Figure 13.D.1(a), the competitive equilibrium with perfect information is sustained as a separating equilibrium:

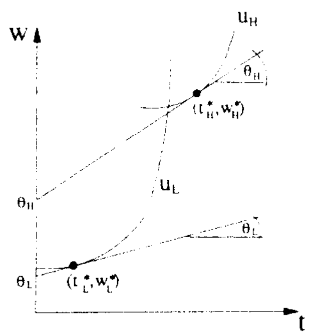


Figure 13.D.1(a)

In Figure 13.D.1(b), the outcome of the competitive equilibrium cannot be

attained as an equilibrium with imperfect information:

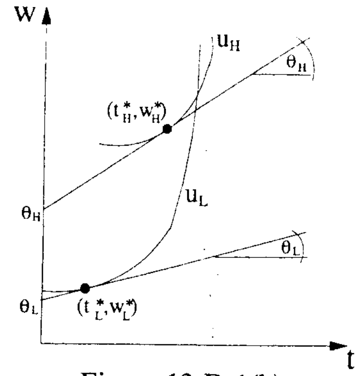


Figure 13.D.1(b)

Type θ_L would like to pretend to be of type θ_H, if he is offered $(t_L^*, \theta_L + \theta_L t_L^*)$ and $(t_H^*, \theta_H + \theta_H t_H^*)$. Thus, the incentive constraint for the low productivity type is not satisfied.

By a similar argument as in the text we can show:

Claim 2: The separating equilibrium of this model exists and is as follows:

- $(e_L^*, \theta_L + \theta_L e_L^*)$ and $(e_H^*, \theta_H + \theta_H e_H^*)$,
- the indifference curve of the high productivity type is above the pooling break-even line.
- no cross-subsidizing contract, that will be accepted by both types, exists that gives a firm a positive profit.

This equilibrium is shown in Figure 13.D.1(c):

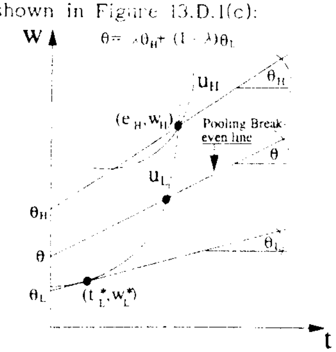


Figure 13.D.1(c)

Claim 4: No equilibrium exists in the following case: The indifference curve of the high productivity type is at some points below the pooling break-even line. This equilibrium is shown in Figure 13.D.1(d):

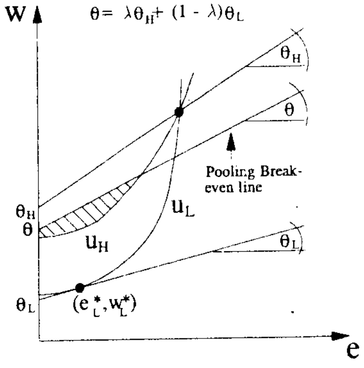


Figure 13.D.1(d)

13.D.2 **(a)** Once M and R are given, the wealth levels of an insured individual in the two states, denoted by (w_1, w_2), are given by:

$$(w_1, w_2) = (W - M,\ W - L - M + R).$$

Therefore, we can think of the insurance contract as specifying the wealth levels (w_1, w_2) in the two states. The premium M and the repayment R can be obtained by the following equations: $M = W - w_1$ and $R = w_2 + L - w_1$.

(b) This game is analogous to the screening game studied in this chapter. Therefore there exists no pooling equilibrium, and the existence of a separating equilibrium is not always assured. If there exists a separating equilibrium, then the high risk types are completely insured, i.e. $w_1^H = w_2^H$. The low risk types will not be completely insured, in fact $w_1^L > w_2^L$.

For a proof of these results, we refer to the original paper by Rothschild, M. and Stiglitz, J.(1977) "Equilibrium in Competitive Insurance Markets with Adverse Selection," QJE, p. 629 - 649 and to Laffont (1989) *The Economics of Uncertainty and Information*, MIT press, Chapter 8.

13.D.3 **(a)** Assume for simplicity that $\mu=1$, which does not change the qualitative results. The high (respectively, low) type workers output is

$\theta_H(1+T)$ (respectively, $\theta_L(1+T)$). A firm can now deduce from the observed output level the worker's type. Therefore, any firm will propose the following contract:

- $\theta_H(1+T)$ if observed output is $\theta_H(1+T)$
- $\theta_L(1+T)$ if observed output is $\theta_L(1+T)$.

If $\theta_L(1+T) \geq c$, both types will accept the offer and work.

If $\theta_H(1+T) \geq c > \theta_L(1+T)$, then only high type worker will accept the contract.

If $c > \theta_H(1+T)$, none of the workers will accept the offer.

(b) Denote by (w_G, w_B) the contract offered by a firm, in which it pays w_G (w_B) if the output realization is good (bad). The firm chooses its contract,

(c) The problem only differs in respect to which workers will accept the offer given in part b):

A worker of high type accepts, if $p_H\, u(w_G) + (1-p_H)\, u(w_B) \geq u(c)$.

A worker of low type accepts, if $p_L u(w_G) + (1-p_L)\, u(w_B) \geq u(c)$.

13.D.4 It can be shown that under conditions (i) and (ii), Proposition 13.D.2 still remains true. However, with these added conditions The model differs from the model of section 13.D. with respect to the existence of equilibria. In this model the existence of an equilibrium is guaranteed if no firm can offer a pooling contract, that attracts both types of workers, and makes a positive profit. Thus, the existence of an equilibrium is guaranteed in Figure 13.D.7(a) in the textbook. In Figure 13.D.7(b), however, there exists no equilibrium, as is the case for the original model. The difference for this model is for the situation displayed in Figure 13.D.8 of the textbook. Contrary to the model discussed in the text, firms cannot offer multiple contracts and thus cannot cross-subsidize between the workers, and an equilibrium will exist.

For a proof of these results, we refer to the original paper by Rothschild, M. and Stiglitz, J.(1977) "Equilibrium in Competitive Insurance Markets with Adverse Selection," QJE, p. 629 - 649 and to Laffont (1989) *The Economics of Uncertainty and Information*, MIT press, Chapter 8.

13.AA.1 An example appears in Section V of:

Cho, I-K., and D.M. Kreps (1987) "Signaling Games and Stable Equilibria," QJE, 102:179-221.

In their paper, Cho and Kreps use the notion of "Riley outcome" to refer to the textbook notion of "best separating equilibrium". The example itself appears under the sub-title "Case B: More than two types," on pages 212-13.

CHAPTER 14

14.B.1 The answer is yes, and the argument is supplied in footnote 8, immediately after Lemma 14.B.1 in the textbook.

14.B.2 Now, the program cannot be split into a minimization program and afterward a maximization program since the principal is risk averse over $\pi - w(\pi)$. Therefore, letting $u(\cdot)$ denote the principal's utility function, the program becomes:

$$\underset{w(\pi)}{\text{Max}} \int u(\pi - w(\pi))f(\pi|e_H)d\pi$$

$$\text{s.t. (i)} \quad \bar{u} - \int v(w(\pi))f(\pi|e_H)d\pi + g(e_H) \le 0$$

$$\text{(ii)} \quad \int v(w(\pi))f(\pi|e_L)d\pi - \int v(w(\pi))f(\pi|e_H)d\pi + g(e_H) + g(e_L) \le 0$$

where constraint (i) is the participation constraint and (ii) is the incentive constraint (assuming that e_H is the desirable action). Letting γ and μ be the Kuhn-Tucker multipliers for (i), and (ii) respectively, the Kuhn-Tucker FOC is:

$$-u'(\pi - w(\pi))f(\pi|e_H) + \gamma v'(w(\pi))f(\pi|e_H) - \mu[f(\pi|e_L) - f(\pi|e_H)]v'(w(\pi)) = 0$$

which in turn yields:

$$\frac{u'(\pi - w(\pi))}{v'(w(\pi))} = \gamma + \mu\left[1 - \frac{f(\pi|e_L)}{f(\pi|e_H)}\right].$$

Note that in this case the incentive constraint may not bind, i.e., we may have $\mu = 0$. The reason is that due to optimal risk sharing it may be optimal for the agent to have enough risk such that (ii) does not bind.

14.B.3 [First Printing Errata: part **(c)** should end with "What effect do changes in ϕ and σ^2 have?"]

(a) Direct calculation gives:

$$\begin{aligned} Eu(w(\pi),e) &= E[\alpha + \beta\pi|e] - \phi VAR[\alpha + \beta\pi|e] - g(e) \\ &= \alpha + \beta E[\pi|e] - \phi\beta^2 VAR[\pi|e] - g(e) \\ &= \alpha + \beta e - \phi\beta^2\sigma^2 - g(e)\ . \end{aligned}$$

(b) Optimal risk sharing will result in a fixed wage for the agent, and maximizing the principal's profits ensures that this wage will exactly compensate the agent for his effort. Therefore, the first-best contract with observable (and verifiable) effort is the solution to:

$$\underset{e}{\text{Max}}\ \ E[\pi|e] - g(e)\ ,$$

and the FOC (which is necessary and sufficient) gives $g'(e^*) = 1$, which gives us the optimal effort level e^*, and $w = g(e^*)$ is the wage.

(c) As in section 14.B of the textbook, the principal's problem can be divided into two steps. First, for a given effort level e', the optimal individually rational and incentive compatible compensation scheme (using the result from part **(a)** above) is given by:

$$\underset{\alpha,\beta}{\text{Min}}\ \ \alpha + \beta e'$$

$$\text{s.t. (i)}\ \ \alpha + \beta e' - \phi\beta^2\sigma^2 - g(e') \geq 0$$

$$\text{(ii)}\ \ \alpha + \beta e' - \phi\beta^2\sigma^2 - g(e') \geq \alpha + \beta e - \phi\beta^2\sigma^2 - g(e)\ \ \forall\ e \neq e'$$

The incentive constraint (ii) implies that given e', $\beta e - g(e)$ should reach a maximum at e'. The condition of the question allow us to replace (ii) with the FOC: $g'(e') = \beta$, which uniquely determines β given e'. We also know that (i) will bind, so $\beta = g'(e')$ implies: $\alpha = g(e') + \phi g'(e')^2\sigma^2 - g'(e')e'$. We

can now find the optimal compensation scheme by solving:

$$\underset{e}{\text{Max}}\ \ E[\pi|e] - E[g(e) + \phi g'(e)^2\sigma^2 - g'(e)e + g'(e)\pi|e]\ ,$$

which reduces to:

$$\underset{e}{\text{Max}}\ \ e - g(e) - \phi g'(e)^2\sigma^2.$$

Given the conditions of the question, this is a concave program, so the FOC which is necessary and sufficient yields: $1 - g'(e) - 2\phi\sigma^2 g'(e)g''(e) = 0$, which gives us: $g'(e) = \dfrac{1}{1 + 2\phi\sigma^2 g'(e)}$. This implies that $0 < \beta < 1$, which is "economically" reasonable because the incentives are not fully aligned with profits due to optimal risk sharing. As ϕ increases, the agent is more averse to risk (through variance) and therefore β will be lower, i.e., lower incentives. The same happens as σ^2 increases.

14.B.4 [First Printing Errata: in the hint of part **(b)** it should read "v_L and v_H" and not "v_1 and v_2"]

(a) Let $p_i \equiv f(\pi_H|e_i)$ so that $p_1 = \frac{2}{3}$, $p_2 = \frac{1}{2}$, and $p_3 = \frac{1}{3}$. When effort is observable, the principal will pay exactly $g(e)$ for effort level e, so that:

$$\pi(e_1) = \frac{2}{3}\cdot 10 + \frac{1}{3}\cdot 0 - \frac{25}{9} > 3\ ,$$

$$\pi(e_2) = \frac{1}{2}\cdot 10 + \frac{1}{2}\cdot 0 - \frac{64}{25} < 3\ ,$$

$$\pi(e_3) = \frac{1}{3}\cdot 10 + \frac{2}{3}\cdot 0 - \frac{16}{9} < 2\ ,$$

and therefore e_1 is optimal with a wage of $w = \frac{25}{9}$.

(b) Let a contract specify a pair (v_H, v_L) where $v_k \equiv v(w_k)$. For e_2 to be implementable, three conditions must hold:

$$\text{(i)}\ \ \frac{1}{2}\cdot v_H + \frac{1}{2}\cdot v_L - \frac{8}{5} \geq 0\ ,$$

$$\text{(ii)}\ \ \frac{1}{2}\cdot v_H + \frac{1}{2}\cdot v_L - \frac{8}{5} \geq \frac{2}{3}\cdot v_H + \frac{1}{3}\cdot v_L - \frac{5}{3}\ ,$$

calculations show that (ii) implies $v_H \leq \frac{2}{5} + v_L$, and (iii) implies $v_H \geq \frac{8}{5} + v_L$, and clearly both cannot be satisfied simultaneously. For e_2 to be implementable we must have both (ii) and (ii) satisfied. Rewriting both:

$$\text{(ii)} \quad \frac{1}{2}\cdot v_H + \frac{1}{2}\cdot v_L - g(e_2) \geq \frac{2}{3}\cdot v_H + \frac{1}{3}\cdot v_L - \frac{5}{3},$$

$$\text{(iii)} \quad \frac{1}{2}\cdot v_H + \frac{1}{2}\cdot v_L - g(e_2) \geq \frac{1}{3}\cdot v_H + \frac{2}{3}\cdot v_L - \frac{4}{3},$$

or,

$$\text{(ii)} \quad 10 - 6\cdot g(e_2) + v_L \geq v_H,$$

$$\text{(iii)} \quad 6\cdot g(e_2) - 8 + v_L \leq v_H.$$

Both can be satisfied if and only if $g(e_2) \leq \frac{3}{2}$.

(c) [First printing errata: the end of this part of the question should read: "What effects do changes in ϕ and σ^2 have?".]

We established in **(b)** above that e_2 cannot be implemented. To implement e_3 we get the first best contract by paying $w = g(e_3) = \frac{14}{9}$, and the principal's expected profit is:

$$\pi(e_3) = \frac{1}{3}\cdot 10 + \frac{2}{3}\cdot 0 - \frac{16}{9} = \frac{14}{9}.$$

To implement e_1 we must have both individual rationality and incentive constraints satisfied. Consider the individual rationality constraint, and the incentive constraint with respect to e_3:

$$\text{(i)} \quad \frac{2}{3}\cdot v_H + \frac{1}{3}\cdot v_L - \frac{5}{3} = 0,$$

$$\text{(ii)} \quad \frac{2}{3}\cdot v_H + \frac{1}{3}\cdot v_L - \frac{5}{3} = \frac{1}{3}\cdot v_H + \frac{2}{3}\cdot v_L - \frac{4}{3},$$

which together give: $(v_H, v_L) = (2,1)$, or in terms of wages: $(w_H, w_L) = (4,1)$. It is easy to check that incentives with respect to e_2 are satisfied. The principal's expected profit is:

$$\pi(e_1) = \frac{2}{3}\cdot 10 + \frac{1}{3}\cdot 0 - \frac{2}{3}\cdot 4 - \frac{1}{3}\cdot 1 = \frac{33}{9},$$

and therefore e_1 is optimal with the compensation scheme $(w_H, w_L) = (4,1)$.

(d) When effort is observable, the principal will pay exactly $g(e)$ for effort level e, so that:

$$\pi(e_1) = x\cdot 10 + (1-x)\cdot 0 - 8 \longrightarrow 2,$$

$$\pi(e_2) = \frac{1}{2}\cdot 10 + \frac{1}{2}\cdot 0 - \frac{64}{25} = \frac{61}{25} > 2,$$

$$\pi(e_3) = \frac{1}{3}\cdot 10 + \frac{2}{3}\cdot 0 - \frac{16}{9} = \frac{14}{9} < 2,$$

and therefore as $x \longrightarrow 1$, e_1 is optimal with a wage of $w = 8$.

When effort is not observable, the principal can still implement e_3 as in **(c)** above with $\pi(e_3) = \frac{14}{9}$. To implement e_2 we must have (similar to **(b)** above):

$$\text{(i)} \quad \frac{1}{2}\cdot v_H + \frac{1}{2}\cdot v_L - \frac{8}{5} \geq 0,$$

$$\text{(ii)} \quad \frac{1}{2}\cdot v_H + \frac{1}{2}\cdot v_L - \frac{8}{5} \geq x\cdot v_H + (1-x)\cdot v_L - \sqrt{8},$$

$$\text{(iii)} \quad \frac{1}{2}\cdot v_H + \frac{1}{2}\cdot v_L - \frac{8}{5} \geq \frac{1}{3}\cdot v_H + \frac{2}{3}\cdot v_L - \frac{4}{3}.$$

Supposing that (i) and (iii) bind, we compute that $(v_H, v_L) = (\frac{12}{5}, \frac{4}{5})$, or in terms of wages: $(w_H, w_L) = (\frac{144}{25}, \frac{16}{25})$, and it is easy to verify that (ii) is satisfied. We therefore get:

$$\pi(e_2) = \frac{1}{2}\cdot[10 - \frac{144}{25}] + \frac{1}{2}\cdot[0 - \frac{16}{25}] = \frac{49}{25} < 2.$$

To implement e_1 the principal can use a scheme so that as x is arbitrarily close to 1, the cost of implementing e_1 will be arbitrarily close to 8, the wage which will exactly compensate the agent for his efforts. To see this, take some number $\delta > 0$, and let $x = 1-\varepsilon$ where $\varepsilon \longrightarrow 0$. Set the utility compensation values $(v_H, v_L) = ([1 + \delta]\cdot\sqrt{8}, [1 + \delta - \frac{\delta}{\varepsilon}]\cdot\sqrt{8})$. It is easy to check that for all ε, given e_1 this scheme gives the agent an expected utility of 0. Furthermore, as $\varepsilon \longrightarrow 0$ we have that $v_L \longrightarrow -\infty$, so that the incentive constraints will easily be satisfied, and as $\varepsilon \longrightarrow 0$ we can make δ arbitrarily small so that both the individual rationality and incentive constraints are satisfied, and the expected wage that the principal will pay will approach 8, resulting in $\pi(e_1) \longrightarrow 2$. therefore, as $\varepsilon \longrightarrow 0$ the principal will prefer to implement e_1, a higher level than the first best level of effort.

14.B.5 Assume this is not true, i.e., that the optimal incentive scheme is as shown in Figure 14.B.5(a) below:

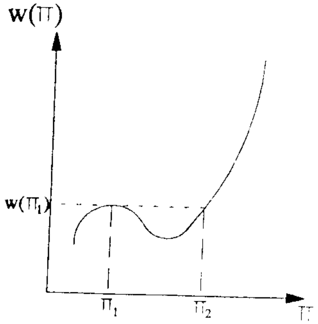


Figure 14.B.5(a)

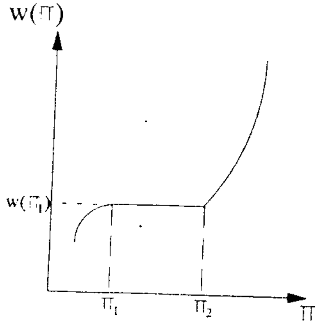


Figure 14.B.5(b)

Then, if any profit $\pi \in (\pi_1,\pi_2)$ was realized, the agent will dispose of any excess profits above π_1, and receive $w(\pi_1) > w(\pi)$. Thus, for all realizations $\pi \in (\pi_1,\pi_2)$ the principal will end up paying $w(\pi_1)$. But this can be achieved by the scheme depicted in Figure 14.B.5(b) above - a contradiction.

14.B.6 (a) The principal's problem becomes:

$$\underset{w(R,C)}{\text{Min}} \int_{\underline{R}}^{\bar{R}}\int_{\underline{C}}^{\bar{C}} w(R,C)f_R(R|e_C)f_C(C|e_C)\, dC\, dR$$

$$\text{s.t.}\quad \text{(i)}\quad \int_{\underline{R}}^{\bar{R}}\int_{\underline{C}}^{\bar{C}} w(R,C)f_R(R|e_C)f_C(C|e_C)\, dC\, dR - g(e_C) \geq 0$$

$$\text{(ii)}\quad \int_{\underline{R}}^{\bar{R}}\int_{\underline{C}}^{\bar{C}} w(R,C)f_R(R|e_C)f_C(C|e_C)\, dC\, dR - g(e_C)$$

$$\geq \int_{\underline{R}}^{\bar{R}}\int_{\underline{C}}^{\bar{C}} w(R,C)f_R(R|e_R)f_C(C|e_R)\, dC\, dR - g(e_R)\ ,$$

where constraint (i) is the participation constraint and (ii) is the incentive constraint (assuming that e_C is the desirable action). Letting γ and μ be the Kuhn-Tucker multipliers for (i) and (ii) respectively, the FOC can be worked out as is done in section 14.B and further algebra yields:

$$\frac{1}{v'(w(R,C))} = \gamma + \mu\cdot\left[1 - \left(\frac{f_R(R|e_R)}{f_R(R|e_C)}\right)\left(\frac{f_C(C|e_R)}{f_C(C|e_C)}\right)\right]$$

As R increases, $\frac{f_R(R|e_R)}{f_R(R|e_C)}$ increases, and the concavity of $v(\cdot)$ implies that $w(\cdot,\cdot)$ should decrease. This makes sense because we want to suppress the incentives of the agent to choose the revenue enhancing effort. Similarly, as C increases, $\frac{f_C(C|e_R)}{f_C(C|e_C)}$ increases, and the concavity of $v(\cdot)$ implies that $w(\cdot,\cdot)$ should decrease. This makes sense because we want to strengthen the incentives of the agent to choose the cost reducing effort.

(b) In this case the principal can no longer use R as a variable in the compensation scheme. The intuition is straightforward: no compensation scheme that induces the agent to choose e_C will have $\frac{\partial w}{\partial R} > 0$, and if $\frac{\partial w}{\partial R} < 0$ for some values of R, the manager will dispose of some revenues. Therefore we must have $\frac{\partial w}{\partial R} = 0$ for all values of R. Thus, the optimal scheme will be $w(C)$ with $\frac{\partial w}{\partial C} < 0$. (The FOC will be as in condition 14.B.10 of the textbook.)

(c) In this case no compensation scheme can induce the manager to exert effort level e_C. That is, only e_R is implementable.

14.B.7 For an analysis of this problem we refer to:

Rogerson, W. (1985) "Repeated Moral Hazard," *Econometrica*, 53:69-76.

It is worth mentioning that the conclusion of this model is counter-intuitive: The optimal compensation scheme is history dependent. The reason is that in addition to supplying the agent with correct incentives, the compensation scheme also serves as a consumption smoothing device for the agent.

14.B.8 For an analysis of this problem we refer to:

Dye, R. (1986) "Optimal Monitoring policies in agencies," *The Rand Journal of Economics*, 17:339-50.

14.C.1 Let $\theta \in \{\theta_1,\theta_2,\ldots,\theta_N\}$, where state θ_i occurs with probability $\lambda_i > 0$, and $\sum_{i=1}^{N}\lambda_i=1$. The revelation principle still holds, so we can restrict ourselves to a menu of contracts of the form $\{(w_i,e_i)\}_{i=1}^{N}$. Program 14.C.8 in the textbook now becomes:

$$\max_{\{(w_i,e_i)\}} \sum_{i=1}^{N} \lambda_i[\pi(e_i) - w_i]$$

$$\text{s.t. (i)} \quad w_i - g(e_i,\theta_i) \geq v^{-1}(\bar{u}) \qquad \forall\, i$$

$$\text{(ii)} \quad w_i - g(e_i,\theta_i) \geq w_j - g(e_j,\theta_j) \;\; \forall\, i \text{ and } \forall\, j \neq i$$

There are N individual rationality (IR) constraints given by (i) above, and (N-1)N incentive compatibility (IC) constraints given by (ii) above. However, an analog of Lemma 14.C.1 implies that only one IR constraint, that of type θ_1, binds. To see this note that the IC constraint ensuring that type θ_2 will not choose (w_1,e_1), the IR constraint of type θ_1, and the assumption that $g_{e\theta}(\cdot,\cdot) < 0$, together imply that:

$$w_2 - g(e_2,\theta_2) \geq w_1 - g(e_1,\theta_2) > w_1 - g(e_1,\theta_1) \geq v^{-1}(\bar{u}) \,,$$

and this can be repeated inductively. Now that we have only one IR constraint, Lemma 14.C.2 holds as before, and we have: $w_1 - g(e_1,\theta_1) = v^{-1}(\bar{u})$.

The analog to Lemma 14.C.3 is that two type of inequalities hold:

$$\text{(I)} \quad e_i \leq e_i^* \,, \quad \text{and,}$$

$$\text{(II)} \quad e_N = e_N^* \,.$$

The analog to Lemma 14.C.4 is that $e_i \leq e_i^*$ for all $i < N$. The proof of this can be seen with a graphical argument as in the textbook extended to N types, or in a similar (yet more cumbersome) way to that of Appendix B in the textbook. Moreover, we will only have the "downward" IC constraints binding, i.e., (ii) above can be replaced with:

$$\text{(ii}'\text{)} \quad w_i - g(e_i,\theta_i) \geq w_{i-1} - g(e_{i-1},\theta_i) \;\; \forall\, i > 1 \,.$$

To see this, note first that for all i > 2 wee can drop the IC constraints with respect to all $j < i - 1$ because:

$$(1) \qquad w_{i-1} - g(e_{i-1},\theta_{i-1}) \geq w_{i-2} - g(e_{i-2},\theta_{i-1}) \,,$$

$$(2) \qquad w_i - g(e_i,\theta_i) \geq w_{i-1} - g(e_{i-1},\theta_i) \,,$$

summing (1) and (2) we get that:

$$w_i - g(e_{i-1},\theta_{i-1}) - g(e_i,\theta_i) \geq w_{i-2} - g(e_{i-2},\theta_{i-1}) - g(e_{i-1},\theta_i).$$

Rewriting this, and using the fact that $g_{e\theta}(\cdot,\cdot) < 0$ we get:

$$w_i - g(e_i,\theta_i) \geq w_{i-2} - g(e_{i-2},\theta_{i-1}) + [g(e_{i-1},\theta_{i-1}) - g(e_{i-1},\theta_i)$$

$$> w_{i-2} - g(e_{i-2},\theta_{i-1})$$

$$> w_{i-2} - g(e_{i-2},\theta_i) \,.$$

To show that all the "upward binding IC constraints are not binding we can follow a similar manner as suggested at the end of Appendix B in the textbook.

14.C.2 The manager's utility function becomes $u(w,e,\theta) = w - g(e,\theta)$. It is easy to show (similar to the textbook analysis where the state θ is observable) that the first best effort level must satisfy $\pi'(e_i^*) = g_e(e_i^*,\theta_i)$

for $i \in \{H,L\}$. Furthermore, when θ is observable the manager has his individual rationality constraint binding, that is:

$$\lambda[w_H - g(e_H^*,\theta_H)] + (1-\lambda)[w_L - g(e_L^*,\theta_L)] = \bar{u} ,$$

or,

$$\lambda w_H + (1-\lambda)w_L = \bar{u} + \lambda g(e_H^*,\theta_H) + (1-\lambda)g(e_L^*,\theta_L) .$$

This, in turn, gives the owner expected profits equal to:

$$E\pi = \lambda[\pi(e_H^*) - w_H] + (1-\lambda)[\pi(e_L^*) - w_L)]$$
$$= \lambda\pi(e_H^*) + (1-\lambda)\pi(e_L^*) - [\bar{u} + \lambda g(e_H^*,\theta_H) + (1-\lambda)g(e_L^*,\theta_L)].$$

(Note that due to risk uncertainty for both the manager and the owner, any pair (w_H,w_L) that satisfies the manager's individual rationality constraint above with equality, will give the owner the same expected profits as above.)

Now suppose that θ is not observable, and the owner offers the manager the following compensation scheme: $w(\pi) = \pi - \alpha$, where

$$\alpha = \lambda\pi(e_H^*) + (1-\lambda)\pi(e_L^*) - w_H] - [\bar{u} + \lambda g(e_H^*,\theta_H) + (1-\lambda)g(e_L^*,\theta_L)].$$

A manager of type i who faces this scheme will choose a level of effort that maximizes $w(\pi(e)) - g(e,\theta_i)$, which is (recall that α is a constant):

$$\underset{e}{\text{Max}} \quad \pi(e) - \alpha - g(e,\theta_i)$$

and the FOC is just the first best condition: $\pi'(e_i^*) = g_e(e_i^*,\theta_i)$. We only need to check that the owner makes the same profit as in the first best scenario, and that the manager will choose to participate. First, since the owner will pay $w(\pi) = \pi - \alpha$ for any realization π, he is left with a profit of α, which is exactly his expected utility in the first best scenario. Second, the manager's expected utility if he accepts the contract is:

$$Eu = \lambda\pi(e_H^*) + (1-\lambda)\pi(e_L^*) - \alpha - \lambda g(e_H^*,\theta_H) - (1-\lambda)g(e_L^*,\theta_L) = \bar{u}.$$

This is shown in the (w,π) space in Figure 14.C.2(a), and in the (w,e) space in Figure 14.C.2(b).

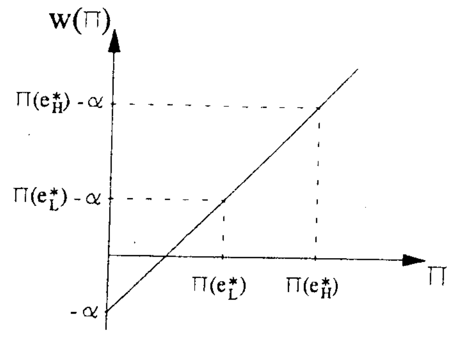


Figure 14.C.2(a)

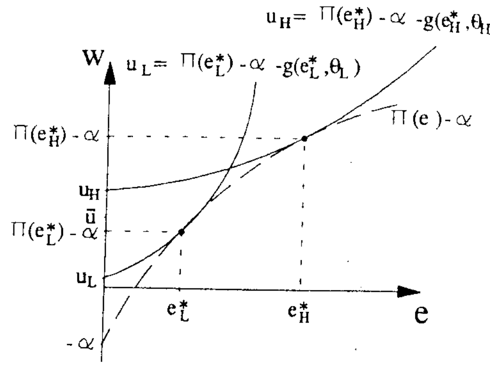


Figure 14.C.2(b)

The revelation mechanism which yields the same outcome is just the two points which result from the compensation scheme above. That is, given α as above, $(w_H,e_H) = (\pi(e_H^*) - \alpha, e_H^*)$, and $(w_L,e_L) = (\pi(e_L^*) - \alpha, e_L^*)$.

14.C.3 (a) Assuming the same conditions on $v(\cdot)$ and $g(\cdot,\cdot)$ the program for the optimal contract under full observability is:

$$\text{Max} \quad \lambda[\pi(e_H) - w_H] + (1-\lambda)[\pi(e_L) - w_L]$$

$$\text{s.t.} \quad \lambda[v(w_H) - g(e_H,\theta_H)] + (1-\lambda)[v(w_L) - g(e_L,\theta_L)] \geq \bar{u} .$$

Letting γ denote the Kuhn-Tucker multiplier, and noting that as in the textbook analysis the assumptions imply that the FOCs must hold with equalities, we get the following four FOCs:

(i) $-\lambda + \gamma[\lambda v'(w_H^*)] = 0$,

(ii) $-(1-\lambda) + \gamma[(1-\lambda)v'(w_L^*)] = 0$,

(iii) $\lambda\pi'(e_H^*) - \gamma\lambda g_e(e_H^*,\theta_H) = 0$,

(iv) $(1-\lambda)\pi'(e_L^*) - \gamma(1-\lambda)g_e(e_L^*,\theta_L) = 0$.

From (i) and (ii) we get that $w_L^* = w_H^*$. This makes sense because wages and

(i) $-\lambda + \gamma[\lambda v'(w^*_H)] = 0$,

(ii) $-(1-\lambda) + \gamma[(1-\lambda)v'(w^*_L)] = 0$,

(iii) $\lambda\pi'(e^*_H) - \gamma\lambda g_e(e^*_H,\theta_H) = 0$,

(iv) $(1-\lambda)\pi'(e^*_L) - \gamma(1-\lambda)g_e(e^*_L,\theta_L) = 0$.

From (i) and (ii) we get that $w^*_L = w^*_H$. This makes sense because wages and disutility of effort are separable now, and the risk aversion is only on monetary income so that an optimal contract will have both wage levels equal. Using this we can rewrite (iii) and (iv), for states H and L respectively, as:

$$\pi'(e^*_i) = \frac{1}{v'(w^*_i)} \cdot g_e(e^*_i,\theta_i) \ .$$

This again makes sense because having $w^*_L = w^*_H$ implies that $e^*_L < e^*_H$.

(b) This contract is clearly not feasible when θ is unobservable. The reason is that $w^*_L = w^*_H = w^*$, and $e^*_L < = e^*_H$, will cause the manager to choose the L pair if the H state occurs, i.e., the H type will misrepresent.

14.C.4 For an analysis of this problem we refer to:

Hart, O. (1983) "Optimal Labor Contracts under Asymmetric Information: An Introduction," *The Review of Economic Studies*, 50:3-35.

14.C.5 [First Printing Errata: the first line of the question should read "... in Section 14.C would not change if..."]

The objective function in (14.C.1) and (14.C.8) in the textbook changes to:

$$\lambda[\pi_H(e_H) - w_H] + (1-\lambda)[\pi_L(e_L) - w_L]$$

and the first best effort levels are determined by: $\pi'_i(e^*_i) = g_e(e^*_i,\theta_i)$ for $i \in \{H,L\}$, and the condition $\pi'_H(e) \geq \pi'_L(e) > 0$ ensures that $e^*_H > e^*_L$, and this promises that the analysis in section 13.C would not change. If, however, we have $0 < \pi'_H(e) < \pi'_L(e)$, it may be that $e^*_H < e^*_L$, and in this case the analysis of section 13.C would no longer apply. (Note that the

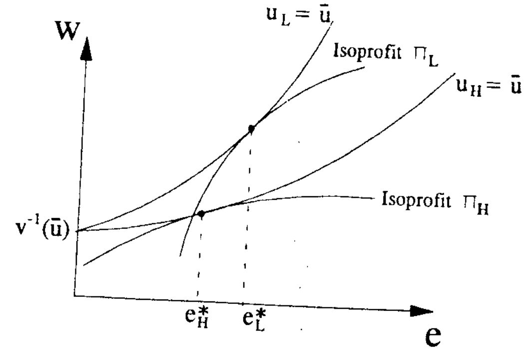


Figure 14.C.5(a)

We present a brief graphical presentation of two possibilities for optimal contracts when $e^*_H < e^*_L$. If the menu $\{(w_H,e_H),(w_L,e_L)\}$ is incentive compatible we must have $e_H > e_L$ (see Figure 14.C.4 in the textbook). Now start from the situation $(w_L,e_L)=(w^*_L,e^*_L)$ where w^*_L is chosen such that $u_L = \bar{u}$. Because$_L > e^*_H$, the best contract for the H type will have $e_H = e_L = e^*_L$ since this has the least distortion of the H type's effort level. This means we will have a pooling contract with $(w,e)=(w^*_L,e^*_L)$. This, however, cannot be optimal because a first order reduction of w and of e along the indifference curve of the L type, $u_L = \bar{u}$, will cause no first order change for the L type yet will cause a first order benefit for the H type both with respect to a lower wage, and a better effort choice (closer to e^*_H) . Therefore, if the probability of θ_H is not too large, we will end up at a pooling contract of the type $(\hat{w},\hat{e})$, where $e^*_H \leq \hat{e} \leq e^*_L$, and $\hat{w}$ is chosen so that the L type's individual rationality constraint is bonding. An illustration of this situation is given in Figure 14.C.5(b). If, on the other hand, λ_H is large enough, then it will be worthwhile to further distort the L type contract so that we get $e_H = e^*_H$, and $e_L < e^*_H$. An illustration of this is given in Figure 14.C.5(c).

(b) In the program above, constraint (i) and (iv), together with $\theta_B < \theta_T$, imply that constraint (ii) is redundant, so it is never binding. Constraint (i) must therefore bind for if it would not, we can reduce P_T and P_B by $\varepsilon > 0$, and all the remaining constraints will still be satisfied. This implies that tourists will be indifferent between buying and not buying a ticket.

(c) Assume that $\{(P_T,W_T),(P_B,W_B)\}$ is an optimal, incentive compatible contract, and assume in negation that $W_B > 0$. Now reduce W_B by $\varepsilon > 0$, and increase P_B by $\frac{\varepsilon}{\theta_B}$ so that the B type's utility does not change, and the firm earns higher profits from the B type. We need to check that the T type will not choose this new compensation package. Indeed,

$$\theta_T P_T + W_T \le \theta_T P_B + W_B = \theta_T(P_B + \frac{\varepsilon}{\theta_T}) + (W_B - \varepsilon) < \theta_T(P_B + \frac{\varepsilon}{\theta_B}) + (W_B - \varepsilon),$$

contradicting that $\{(P_T,W_T),(P_B,W_B)\}$ is an optimal, incentive compatible contract. Therefore, we must have $W_B = 0$. If, in an optimal contract, the business travelers were not indifferent between (P_T,W_T) and (P_B,W_B), we could slightly raise P_B and all the constraints would remain satisfied (recall that (ii) is redundant), and the firm would earn higher profits from the business types. Therefore, in an optimal contract we must have the business types indifferent between (P_T,W_T) and (P_B,W_B).

(d) The trade of that the firm faces is: By lowering P_T and increasing W_T so as to keep the tourists indifferent between buying a ticket or not, the firm can increase P_B (recall that $W_B = 0$). From parts (b) and (c) above, we can conclude that if the firm raises W_T by ε and lowers P_T by $\frac{\varepsilon}{\theta_T}$ so that the tourists remain indifferent between buying or not, then to keep the business types indifferent between their package and the new tourist package, it can increase P_B by $\frac{\varepsilon(\theta_T-\theta_B)}{\theta_T\theta_B}$. Since this trade off is linear, it is true no matter where we are in the (P,W) space, and therefore it will be profitable if

and only if:

$$\lambda \cdot \frac{\varepsilon}{\theta_T} < (1 - \lambda)\cdot\frac{\varepsilon(\theta_T-\theta_B)}{\theta_T\theta_B},$$

or,

$$\frac{\lambda}{1 - \lambda} < \frac{\theta_T - \theta_B}{\theta_B}.$$

Note that this is independent of the cost c, because this is a revenue trade off (the costs are the same for both types). Therefore, the price discrimination scheme can take on two forms:

(1) If $\frac{\lambda}{1 - \lambda} < \frac{\theta_T - \theta_B}{\theta_B}$, then only the high types will be served and the scheme will be (this assumes that $c < \frac{v}{\theta_B}$):

$$\{(P_T,W_T),(P_B,W_B)\} = \{(0,v),(\frac{v}{\theta_B},0)\}$$

(2) If $\frac{\lambda}{1 - \lambda} > \frac{\theta_T - \theta_B}{\theta_B}$, then both types will be served and the scheme will be a pooling scheme with (this assumes that $c < \frac{v}{\theta_T}$):

$$(P_T,W_T) = (P_B,W_B) = (\frac{v}{\theta_T},0).$$

Since the direction of the inequality in the condition above determines the type of scheme, it is easy to see how changes in λ, θ_T, and θ_B will affect the scheme: If the proportion of B types is large enough (λ small enough) the firm will choose to serve only the B types. If the B types suffer less from prices (θ_B is smaller) then the firm is more likely to serve only them. If the T types suffer less from prices (θ_T is smaller) then the firm is more likely to serve them as well as the B types. Changes in the cost c are discussed in part (e) below.

(e) As long as $c < \frac{v}{\theta_B}$, and we are in case (1) as described in part (d) above, the firm will decide to serve only the business types. If, however, we are in case (2) above, and $\frac{v}{\theta_T} < c < \frac{v}{\theta_B}$, then the scheme described in (d) above cannot be optimal because the firm is losing money. In such a case, the firm will choose the scheme described in case (1) of part (d), and serve only the business types. If $c > \frac{v}{\theta_B}$ the firm will choose not to operate at all.

14.C.9 **(a)** The monopolist will offer the individual a policy that fully insures him (optimal risk sharing) and keeps the individual at the same level of expected utility. That is, if we define $\bar{u} \equiv \theta u(W - L) + (1 - \theta)u(W)$, then the optimal insurance policy has $c_1 = c_2 = u^{-1}(\bar{u})$.

(b) The monopolist will offer an optimal screening contract of the form $\{(c_1^L, c_2^L), (c_1^H, c_2^H)\}$ that solves:

$$\text{Max} \quad \lambda[(1 - \theta_H)c_1^H + \theta_H c_2^H] + (1 - \lambda)[(1 - \theta_L)c_1^L + \theta_L c_2^L]$$

$$\text{s.t.} \quad \text{(i)} \quad (1 - \theta_H)u(c_1^H) + \theta_H u(c_2^H) \geq \bar{u}$$

$$\text{(ii)} \quad (1 - \theta_L)u(c_1^L) + \theta_L u(c_2^L) \geq \bar{u}$$

$$\text{(iii)} \quad (1 - \theta_H)u(c_1^H) + \theta_H u(c_2^H) \geq (1 - \theta_H)u(c_1^L) + \theta_H u(c_2^L)$$

$$\text{(iv)} \quad (1 - \theta_L)u(c_1^L) + \theta_L u(c_2^L) \geq (1 - \theta_L)u(c_1^H) + \theta_L u(c_2^H)$$

This is again a standard monopolistic screening problem and the standard analysis applies. Solving this program (again, (ii) and (iii) will be redundant) will have the H type fully insured, and the L type not fully insured. A graphical analysis is given using Figure 14.C.9 below. Both types start at the point A, with utility levels $\bar{u}_H$ and $\bar{u}_L$ respectively. If the monopolist would try to insure both types by offering the points B and C to H and L respectively, the (unobservable) H type would choose policy C instead of B, that is, the points B and C cannot be part of an incentive compatible contract.

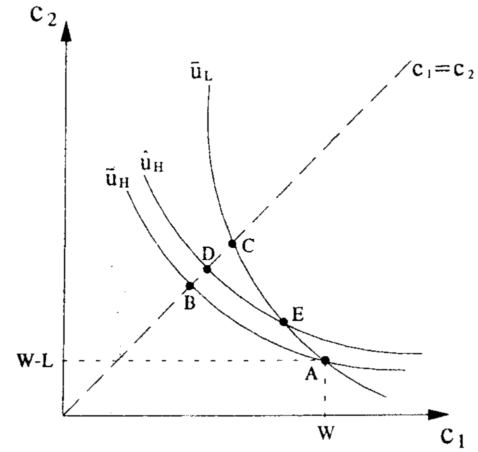


Figure 14.C.9

If the monopolist offers points A and B, then the H types would choose B (since they are indifferent), they would be fully insured, and the risk neutral monopolist would make profits from their choice. The L types, however, would prefer the point A to B and no profits would be made from them. If the proportion of High types is large enough, then the monopolist will find it profitable to slightly insure the L type at the cost of raising the utility of the H type. This means that the optimal contract will look like the two points D and E for the H and L types respectively. The mathematical analysis is straightforward, and results in a situation common to monopolistic screening (or hidden information): The H type will be at the first best insurance level and his participation constraint is not binding. The L type will be under insured (second best distortion so that screening is possible and profitable), his participation constraint will bind and his incentive compatibility constraint will not. (Note, that the pair of points A,B may be optimal if the proportion of L types is small. This is parallel to the hidden information case where the H type is at the first-best observable point with no surplus, and the L type has $e_L = 0$).

(c) The difference is who gets the surplus. In chapter 13 we discussed competitive markets, so that the insurer was left with zero profits and the individuals had utility levels above their reservation utility. Here, the monopolist makes positive profits and at least the L type has no surplus.

14.AA.1 For an analysis of this problem we refer to Proposition 5 in:

Milgrom, P. (1981) "Good News and Bad News: Representation Theorems and Applications," *Bell Journal of Economics*, 12:380-91.

14.AA.2 Sufficient conditions for the first order approach to be valid will promise that the agent's optimization program yields a unique solution. Given a compensation scheme (w_H, w_L), where w_i is the compensation when profits π_i are observed, the agent maximizes:

$$\underset{e}{\text{Max}} \quad f(\pi_H|e)\cdot v(w_H) + [1 - f(\pi_H|e)]\cdot v(w_L) - g(e) .$$

The FOC will be sufficient if the SOC is satisfied, i.e., if

$$f_{ee}(\pi_H|e)\cdot[v(w_H) - v(w_L)] - g''(e) < 0$$

So, if $v(w_H) - v(w_L) > 0$, and $f_{ee}(\pi_H|e) < 0$, then the first order approach will be valid. The first inequality is implies by MLRP, and the second is guaranteed by concavity of the density function. For more on this we refer to:

Rogerson, W. (1985) "The first-order Approach to Principal-Agent Problems," *Econometrica*, 53:1357-69.

14.AA.3 The program to be solved is:

$$\underset{\{(w_H,e_H),(w_L,e_L)\}}{\text{Max}} \lambda[\pi(e_H) - w_H] + (1 - \lambda)[\pi(e_L) - w_L]$$

$$\begin{aligned} \text{s.t. } &\text{(i)} \quad w_L - g(e_L,\theta_L) \geq 0 \\ &\text{(ii)} \quad w_H - g(e_H,\theta_H) \geq 0 \\ &\text{(iii)} \quad w_H - g(e_H,\theta_H) \geq w_L - g(e_L,\theta_H) \\ &\text{(iv)} \quad w_L - g(e_L,\theta_L) \geq w_H - g(e_H,\theta_L) \end{aligned}$$

and we have already established in the textbook that (ii) is redundant (Lemma 14.C.1). We proceed with two straightforward claims for the program without constraint (ii):

Claim 1: Constraint (i) must bind at a solution.

Proof: Assume not. Then reduce both w_L and w_H by $\varepsilon > 0$ such that (i) is still satisfied. This will not change the remaining two constraints and it will increase the objective function, a contradiction to being at an optimum. □

Claim 2: Constraint (iii) must bind at a solution.

Proof: Assume not. Then reduce w_H by $\varepsilon > 0$ such that (iii) is still satisfied. This will not affect constraint (i), constraint (iv) will only be further relaxed, and it will increase the objective function, a contradiction to being at an optimum. □

We now proceed to solve the "modified program" which is the original program without constraints (ii) and (iv), and with constraints (i) and (iii) binding. We will then proceed to show that at the solution to the modified program, constraint (iv) will be satisfied. The modified program is therefore:

$$\underset{\{(w_H,e_H),(w_L,e_L)\}}{\text{Max}} \lambda[\pi(e_H) - w_H] + (1 - \lambda)[\pi(e_L) - w_L]$$

$$\begin{aligned} \text{s.t. } &\text{(i)} \quad w_L - g(e_L,\theta_L) = 0 \\ &\text{(iii)} \quad w_H - g(e_H,\theta_H) = w_L - g(e_L,\theta_H) \end{aligned}$$

Letting γ and μ be the Lagrange multipliers for constraints (i) and (iii) respectively. Then, assuming an interior solution, we get the FOCs:

(1) $\lambda\pi'(e_H) - \mu g_e(e_H,\theta_H) = 0$

(2) $(1 - \lambda)\pi'(e_L) - \gamma g_e(e_L,\theta_L) + \mu g_e(e_L,\theta_H) = 0$

(3) $-\lambda + \mu = 0$

(4) $-(1 - \lambda) + \gamma - \mu = 0$

From (3) we have that $\mu = \lambda$, and plugging this into (4) we get that $\gamma = 1$. Substituting for μ and λ in (1) and (2) we get the well known conditions:

$$\pi'(e_H) = g_e(e_H,\theta_H) ,$$

$$\pi'(e_L) = g_e(e_L,\theta_L) + \frac{\lambda}{1 - \lambda}\cdot[g_e(e_L,\theta_H) - g_e(e_L,\theta_H)] .$$

w_L and w_H are then computed using the two binding constraints. Denote the solution to the modified program by $\{(\hat{w}_H,\hat{e}_H),(\hat{w}_L,\hat{e}_L)\}$. We are left to show that this solution satisfies constraint (iv). We proceed with tow claims regarding the modified program.

<u>Claim 3:</u> At the solution to the modified program: $\pi(\hat{e}_H) - \hat{w}_H \geq \pi(\hat{e}_L) - \hat{w}_L$.

<u>Proof:</u> Assume not. Then the firm can offer the pair $(\hat{w}_L,\hat{e}_L)$ to both types. The L type will clearly accept it, and since we have shown that (iii) is binding then the H type will be indifferent between this pair and his original pair $(\hat{w}_H,\hat{e}_H)$. This, however, will raise the profits earned from the H type while leaving the profits earned from the L type unchanged, a contradiction to $\{(\hat{w}_H,\hat{e}_H),(\hat{w}_L,\hat{e}_L)\}$ being a solution to the modified program. □

<u>Claim 4:</u> Constraint (iv) must be satisfied at a solution to the modified program.

<u>Proof:</u> Assume not. That is, assume that $\hat{w}_L - g(\hat{e}_L,\theta_L) < \hat{w}_H - g(\hat{e}_H,\theta_L)$. Then, the firm can offer $(\hat{w}_H,\hat{e}_H)$ to both types, the H type will clearly accept as he did before, and the low type will prefer this

to $(\hat{w}_L,\hat{e}_L)$ given our negation assumption. Furthermore, $(\hat{w}_H,\hat{e}_H)$ must satisfy (i) because $(\hat{w}_L,\hat{e}_L)$ did and the L type prefers $(\hat{w}_H,\hat{e}_H)$ to $(\hat{w}_L,\hat{e}_L)$. But now the firm is earning profits of $\pi(\hat{e}_H) - \hat{w}_H$ from both types, which is (from Claim 3) at least as good as the profits earned from the L type through $(\hat{w}_L,\hat{e}_L)$. The assumptions on $\pi(\cdot)$ and $g(\cdot,\cdot)$ guarantee a unique solution to the modified program, a contradiction. Therefore, constraint (iv) must be satisfied at a solution to the modified program. □

CHAPTER 15

15.B.1 **(a)** The budget constraint implies that

$$p_1(x_{1i}(p) - \omega_{1i}) + p_2(x_{2i}(p) - \omega_{2i}) \le 0$$

for each i = 1,2. Suppose that the above weak inequality ≤ held with strict inequality <, then there would be $(x_{1i},x_{2i}) \in B_i(p)$ such that $(x_{1i},x_{2i}) \succ_i (x_{1i}(p),x_{2i}(p))$, because the preference of each consumer is locally nonsatiated. This would contradict the fact that $(x_{1i}(p),x_{2i}(p))$ is the demand at p. We must thus have

$$p_1(x_{1i}(p) - \omega_{1i}) + p_2(x_{2i}(p) - \omega_{2i}) = 0$$

for each i = 1,2. Summing over i = 1,2, we obtain

$$p_1(\textstyle\sum_i x_{i1}(p) - \bar{\omega}_1) + p_2(\textstyle\sum_i x_{2i}(p) - \bar{\omega}_2) = 0.$$

(b) If the market for good 1 clears at p*, then $\sum_i x_{1i}(p^*) - \bar{\omega}_1 = 0$ and hence $p_2(\sum_i x_{2i}(p^*) - \bar{\omega}_2) = 0$ by the equality of (a). By $p_2^* > 0$, this implies that $\sum_i x_{2i}(p^*) - \bar{\omega}_2 = 0$. Hence the market for good 2 clears at p* as well and p* is a Walrasian equilibrium price vector.

15.B.2 As we saw in Example 15.B.1, the offer curves of the two consumers are given by

$$OC_1(p) = (\alpha p\cdot\omega_1/p_1,\ (1 - \alpha)p\cdot\omega_1/p_2),$$

$$OC_2(p) = (\beta p\cdot\omega_2/p_1,\ (1 - \beta)p\cdot\omega_2/p_2).$$

The total demand for good 2 at p is thus

$$(p_1/p_2)((1 - \alpha)\omega_{11} + (1 - \beta)\omega_{12}) + ((1 - \alpha)\omega_{21} + (1 - \beta)\omega_{22}).$$

By (b) of Exercise 15.B.1, equating this to $\bar{\omega}_2 = \omega_{21} + \omega_{22}$ gives the equilibrium price ratio $p_1^*/p_2^* = \dfrac{\alpha\omega_{21} + \beta\omega_{22}}{(1 - \alpha)\omega_{11} + (1 - \beta)\omega_{12}}$. By substituting this into the offer curves, we obtain the equilibrium allocations: $OC_1(p^*)$ is equal to

$(\omega_{11}\omega_{21} + \beta\omega_{11}\omega_{22} + (1 - \beta)\omega_{21}\omega_{12})\left(\frac{\alpha}{\alpha\omega_{21} + \beta\omega_{22}}, \frac{1 - \alpha}{(1 - \alpha)\omega_{11} + (1 - \beta)\omega_{12}}\right),$

and $OC_2(p^*)$ is equal to

$$(\omega_{12}\omega_{22} + (1 - \alpha)\omega_{11}\omega_{22} + \alpha\omega_{21}\omega_{12})\left(\frac{\beta}{\alpha\omega_{21} + \beta\omega_{22}}, \frac{1 - \beta}{(1 - \alpha)\omega_{11} + (1 - \beta)\omega_{12}}\right).$$

It is easy to check that

$$\partial(p_1^*/p_2^*)/\partial\omega_{11} < 0,$$

$$\partial OC_{11}(p^*)/\partial\omega_{11} > 0,$$

$$\partial OC_{21}(p^*)/\partial\omega_{11} = \frac{(1 - \alpha)(1 - \beta)\omega_{12}}{\alpha\omega_{21} + \beta\omega_{22}} > 0,$$

$$\partial OC_{12}(p^*)/\partial\omega_{11} > 0,$$

and, since $\partial OC_{21}(p^*)/\partial\omega_{11} > 0$ and $\bar{\omega}_2$ is constant,

$$\partial OC_{22}(p^*)/\partial\omega_{11} < 0.$$

15.B.3 Let p^* be a Walrasian equilibrium price vector and x_i^* be the demand of consumer i (i = 1,2). Since the preference of consumer 1 is locally non-satiated, the upper contour sets $\{x_1 \in \mathbb{R}_+^2: x_1 \succsim_1 x_1^*\}$ lies on or above the budget line and the strict upper contour sets $\{x_1 \in \mathbb{R}_+^2: x_1 \succ_1 x_1^*\}$ lies strictly above the budget line. Symmetrically, the upper contour sets $\{x_2 \in \mathbb{R}_+^2: x_2 \succsim_2 x_2^*\}$ lies on or below the budget line and the strict upper contour sets $\{x_2 \in \mathbb{R}_+^2: x_2 \succ_2 x_2^*\}$ lies strictly above the budget line. Hence the two sets $\{x_1 \in \mathbb{R}_+^2: x_1 \succsim_1 x_1^*\}$ and $\{x_2 \in \mathbb{R}_+^2: x_2 \succ_2 x_2^*\}$ do not intersect; the other two, $\{x_1 \in \mathbb{R}_+^2: x_1 \succ_1 x_1^*\}$ and $\{x_2 \in \mathbb{R}_+^2: x_2 \succsim_2 x_2^*\}$, do not either. Hence the Walrasian equilibrium allocation $x^* = (x_1^*, x_2^*)$ is Pareto optimal. See the figure below.

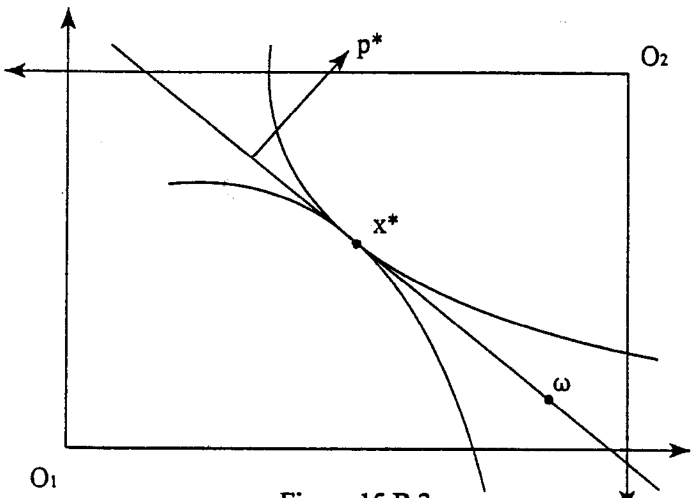


Figure 15.B.3

15.B.4 (a) Here is an example of an offer curve (of consumer 1) with the gross substitute property. The dotted curve is the indifference curve of consumer 1 that goes through ω.

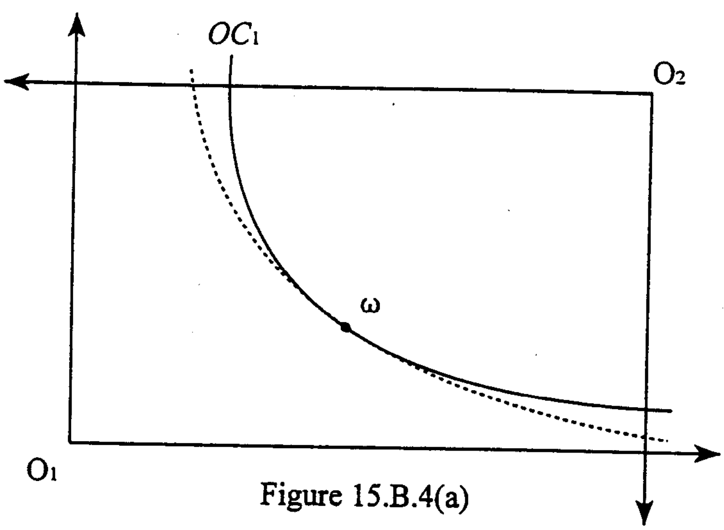
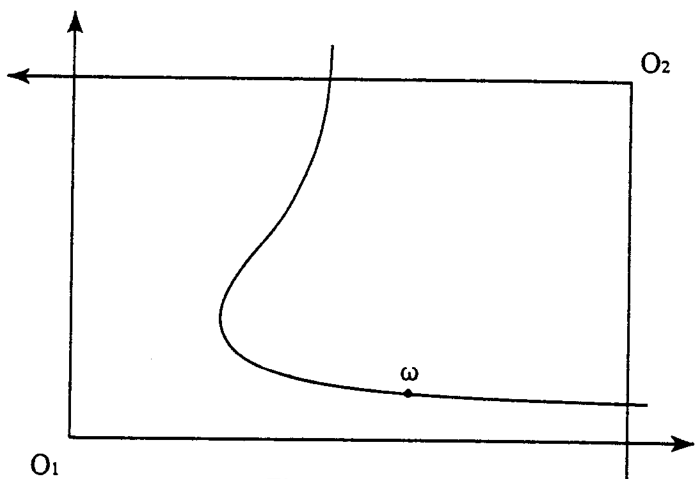


Figure 15.B.4(a)

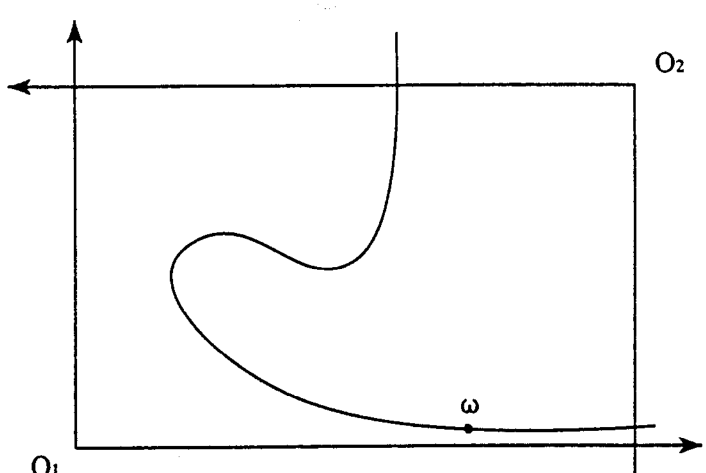
(b) As we discussed in the text, at any equilibrium, the offer curves of the two consumers intersect and, conversely, any intersection of the offer curves at an allocation different from ω corresponds an equilibrium. In order to verify that the offer curves intersect only once (not counting the intersection at the initial endowments), it is therefore sufficient to show that there is only one equilibrium price ratio p_1^*/p_2^*. So, let $p^* = (p_1^*,1)$ be an equilibrium price vector . If $p_1 > p_1^*$, then $OC_{11}(p_1,1) < OC_{11}(p_1^*,1)$ and $OC_{12}(p_1,1) < OC_{12}(p_1^*,1)$ by the gross substitute property. Hence $OC_{11}(p_1,1) + OC_{12}(p_1,1) < OC_{11}(p_1^*,1) + OC_{12}(p_1^*,1) = \bar{\omega}_1$. Hence $(p_1,1)$ is not an equilibrium price vector. Symmetrically, if $p_1 < p_1^*$, then there $(P_1,1)$ is not an equilibrium price vector either. Thus the offer curves intersect only once.

(c) Here is an example of a normal offer curve that does not satisfy the gross substitute property.



Figrue 15.B.4(c)

(d) Here is an example of a preference, an initial endowment, and the corresponding offer curve that is not normal.



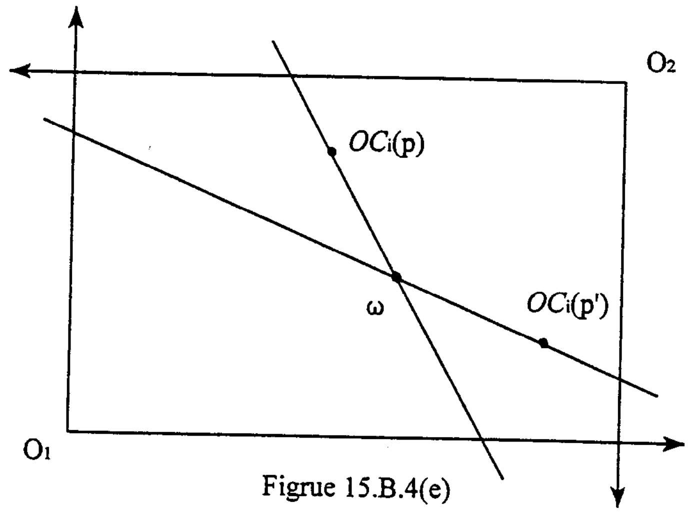
Figrue 15.B.4(d.1)

As for the second statement, we consider the case in which the price of commodity 1 increases. (The case in which the price of commodity 1 increases can be symmetrically proved.) Let (p_1,p_2) be the initial price vector and $p_1' > p_1$. Assume that $OC_{1i}(p_1',p_2) > OC_{1i}(p_1,p_2)$ and $OC_{2i}(p_1',p_2) < OC_{2i}(p_1,p_2)$. It is sufficient to prove that if commodity 1 is not inferior, then commodity 2 must be inferior. Suppose so. Then, by $p_1' > p_1$ and $OC_{1i}(p_1',p_2) > OC_{1i}(p_1,p_2)$, the real wealth must have increased from p_1 to p_1'. (That is, $p_1'OC_{1i}(p_1,p_2) + p_2OC_{2i}(p_1,p_2) < p_1'\omega_{1i} + p_2\omega_{2i}$). Since the relative price of commodity 2 has decreased, this and $OC_{2i}(p_1',p_2) < OC_{2i}(p_1,p_2)$ together imply that commodity 2 must be inferior. This completes the proof.

(e) Assume that the offer curve of consumer 1 is normal and that of consumer 2 satisfies the gross substitute property. If the initial endowments (ω_1,ω_2) constitute a Walrasian equilibrium (and preferences are strictly convex), then the two offer curves intersect only at the initial endowments, because they are contained in the upper contour sets of ω_1 and ω_2. So suppose that the initial endowments do not constitute a Walrasian equilibrium. Then we first need to establish the following assertion:

If both p and p' are equilibrium price vectors, then, for every $i = 1,2$ and every $\ell = 1,2$, we have $(OC_{\ell i}(p) - \omega_{\ell i})(OC_{\ell i}(p') - \omega_{\ell i}) > 0$.

In fact, suppose that we have $(OC_{\ell i}(p) - \omega_{\ell i})(OC_{\ell i}(p') - \omega_{\ell i}) \leq 0$. Since the initial endowments do not constitute an equilibrium, the above weak inequality is satisfied with strict inequality. Thus one of $OC_{\ell i}(p) - \omega_{\ell i}$ and $OC_{\ell i}(p') - \omega_{\ell i}$ must be positive and the other must be negative. By the weak axiom of revealed preference, $OC_i(p)$ and $OC_i(p')$ must be outside the other budget constraints, as shown in the figure below:



Figrue 15.B.4(e)

In the Edgeworth box, however, this implies that the offer curve of the other consumer does not satisfy the weak axiom. This is a contradiction. Hence, for every ℓ and i, we must have

$$(OC_{\ell i}(p) - \omega_{\ell i})(OC_{\ell i}(p') - \omega_{\ell i}) > 0.$$

Therefore, by relabelling the indexes of the commodities if necessary, we can assume that $OC_{11}(p) - \omega_{11} > 0$ for every equilibrium price vector p. Now, let $p^* = (p_1^*,1)$ be an equilibrium price vector such that if $p = (p_1,1)$ is any other equilibrium price vector, then $p_1^* < p_1$. For any $p = (p_1,1)$ with $p_1^* < p_1$, by normality, we have either

$$OC_{11}(p^*) > OC_{11}(p)$$

or

$$OC_{11}(p^*) < OC_{11}(p) \text{ and } OC_{21}(p^*) < OC_{21}(p).$$

Since $OC_{11}(p^*) - \omega_{11} > 0$ and $p_1^* < p_1$, if the second case applies, then $p_1(OC_{11}(p) - \omega_{1i}) + OC_{21}(p) - \omega_{2i} > 0$, which is a contradiction to the budget constraint. Thus the second case is actually impossible and the first case applies. On the other hand, by the gross substitute property, we have $OC_{12}(p^*) > OC_{12}(p)$. Hence $\omega_1 = OC_{11}(p^*) + OC_{12}(p^*) > OC_{11}(p) + OC_{12}(p)$. Thus p is not an equilibrium price vector. Thus the Walrasian equilibrium price vector p* is unique. Therefore, the two offer curves intersect only once, except for the initial endowments.

(f) Here is an example in which two normal offer curves intersect several times.

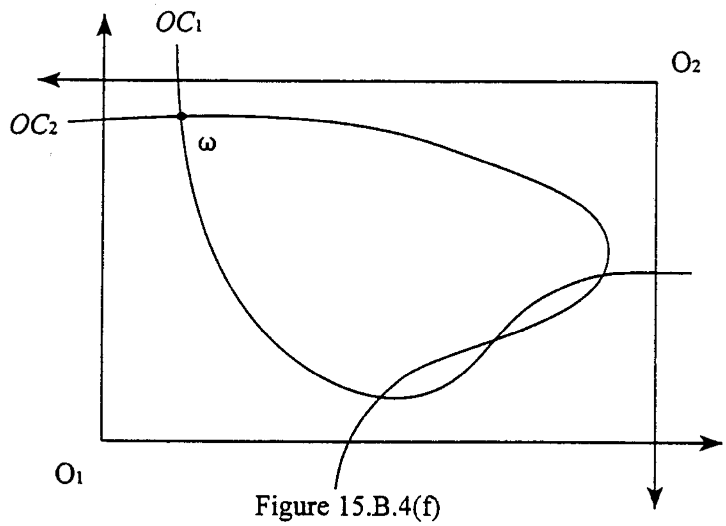


Figure 15.B.4(f)

15.B.5 First, we can derive from the quasilinearity that $OC_{21}(p_1,p_2)^{-9} = p_2/p_1$, that is, $OC_{21}(p_1,p_2) = (p_2/p_1)^{-1/9}$. The budget constraint then implies that $OC_{11}(p_1,p_2) = 2 + r(p_2/p_1) - (p_2/p_1)^{8/9}$. Hence the offer curve of consumer 1 is as given in Example 15.B.2. We can similarly show that the offer curve of consumer 2 is also as given in the example. Rearranging the equality of the total demand of the second good and its total supply and substituting $r = 2^{8/9} - 2^{1/9}$, we obtain

$$(p_1/p_2)^{8/9} - (p_1/p_2)^{1/9} = (2^{8/9} - 2^{1/9})(p_1/p_2 - 1).$$

Then $p_1/p_2 = 1/2, 1, 2$ are solutions of this equation, and hence equilibrium price ratios.

15.B.6 We can obtain the offer curves from the first-order conditions of the utility maximization problem:

$$OC_1(p) = \frac{p_1}{p_1^{2/3} + (12/37)p_2^{2/3}}(p_1^{-1/3}, (12/37)p_2^{-1/3}),$$

$$OC_2(p) = \frac{p_2}{(12/37)p_1^{2/3} + p_2^{2/3}}((12/37)p_1^{-1/3}, p_2^{-1/3}).$$

Set $p_2 = 1$ and write $q = p_1^{1/3}$, then

$$OC_{11}(p) + OC_{12}(p) = \frac{q^2}{q^2 + 12/37} + \frac{(12/37)q^{-1}}{(12/37)q^2 + 1}$$

$$= \frac{q^5 + (37/12)q^3 + q^2 + 12/37}{q^5 + (12/37 + 37/12)q^3 + q}.$$

We can check that $OC_{11}(p) + OC_{12}(p) = 1$ if and only if

$$12q^3 - 37q^2 + 37q - 12 = (q - 1)(4q - 3)(3q - 4) = 0.$$

Thus $q = 1, 3/4, 4/3$. Hence the equilibrium price ratios are given by

$$p_1^*/p_2^* = 1, (3/4)^{1/3}, (4/3)^{1/3}.$$

15.B.7 We shall prove that the set of Pareto optimal allocation looks like a curve in the Edgeworth box. More precisely, we show that there is a one-to-one, continuous mapping from a (non-degenerated) bounded closed interval of $\mathbb{R}$ into the Edgeworth whose image is equal to the Pareto set. This is sufficient for the first assertion of the exercise. For each $i = 1,2$, let $u_i(\cdot)$ be a utility function of consumer i. It is continuous, strongly monotone, and strictly quasiconcave.

Note first that, since $u_2(\cdot)$ is continuous, the set $\{u_2(x_2) \in \mathbb{R}: 0 \le x_2 \le \bar{\omega}\}$ is a (non-degenerated) closed bounded interval. Denote it by $[\delta_0,\delta_1]$. For each $\delta \in [\delta_0,\delta_1]$, consider maximizing $u_1(x_1)$ under the constraints $0 \le x_1 \le \bar{\omega}$ and $u_2(\bar{\omega} - x_1) \ge \delta$. This maximization problem is feasible and, by the compactness of $\{x_1: 0 \le x_1 \le \bar{\omega}, u_2(\bar{\omega} - x_1) \ge \delta\}$, there is at least one solution. We shall now prove that the strict quasiconcavity of the $u_i(\cdot)$ implies that such a solution is unique. Let x_1 and x_1' be distinct solutions, then $u_1((1/2)x_1 + (1/2)x_1')) > u_1(x_1) = u_1(x_1')$. We can assume without loss of generality that $u_2(\bar{\omega} - x_1) \ge u_2(\bar{\omega} - x_1') \ge \delta$. Then, by continuity and strong monotonicity, there exists a unique $\lambda \in [0,1]$ such that $u_2(\lambda(\bar{\omega} - x_1)) = u_2(\bar{\omega} - x'_1)$. By $x_1 \neq x'_1$ and $u_1(x_1) = u_1(x'_1)$, we have $\lambda(\bar{\omega} - x_1) \neq \bar{\omega} - x'_1$. Hence

$$u_2((1/2)\lambda(\bar{\omega} - x_1) + (1/2)(\bar{\omega} - x'_1)) > u_2(\lambda(\bar{\omega} - x_1)) = u_2(\bar{\omega} - x'_1).$$

Thus $u_2(\bar{\omega} - ((1/2)x_1 + (1/2)x'_1)) > \delta$. Therefore $(1/2)x_1 + (1/2)x'_1$ is feasible and attains a higher utility than x_1 and x'_1, a contradiction. Thus there must be a unique solution, which we denote by $\varphi_1(\delta) \in \mathbb{R}^2_+$. Define the mapping φ: $[\delta_0,\delta_1] \to \{(x_1,x_2) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+: x_1 + x_2 = \bar{\omega}\}$ by

$$\varphi(\delta) = (\varphi_1(\delta), \bar{\omega} - \varphi_1(\delta)).$$

It is now sufficient to prove that $\varphi(\cdot)$ is one-to-one and continuous, and its image is equal to the Pareto set. The equality to the Pareto set follows from its construction. As for being one-to-one, note that $u_2(\bar{\omega} - \varphi_1(\delta)) = \delta$; otherwise, by strong monotonicity, a small transfer collinear with $\bar{\omega} - \varphi_1(\delta)$ from consumer 2 to 1 would increase the utility level of consumer 1, a contradiction. It thus remains to verify the continuity. For this, it suffices to prove that $\varphi_1(\cdot)$ is continuous. Let $\{\delta^n\}$ be a sequence in $[\delta_0,\delta_1]$ converging to δ. We shall prove that if $\varphi_1(\delta^n) \to x_1$, then $x_1 = \varphi_1(\delta)$. Note first that, by continuity, $u_2(\bar{\omega} - x_1) \geq \delta$ and hence $u_1(x_1) \leq u_1(\varphi_1(\delta))$. By strong monotonicity, for any sufficiently large n, we can find x_1^n such that $u_2(\bar{\omega} - x_1^n) \geq \delta^n$ and $x_1^n \to \varphi_1(\delta)$. Thus $u_1(x_1^n) \leq u_1(\varphi_1(\delta^n))$. Hence $u_1(\varphi_1(\delta)) \leq u_1(x_1)$. Thus $u_1(\varphi_1(\delta)) = u_1(x_1)$. By $u_2(\bar{\omega} - x_1) \geq \delta$, we obtain $x_1 = \varphi_1(\delta)$.

For the second assertion of the exercise, it is sufficient to prove that if the preferences of the consumers are homothetic and the Pareto set ever cuts the diagonal in the interior of the Edgeworth box, then the Pareto set must coincide with the diagonal. Let $(\bar{\delta}\bar{\omega}, (1 - \bar{\delta})\bar{\omega})$ $(\bar{\delta} \in (0,1))$ a Pareto optimal allocation on the diagonal. By the definition of a homothetic preference (Definition 3.B.6), $(\delta\bar{\omega}, (1 - \delta)\bar{\omega})$ is Pareto optimal for every $\delta \in$

$(0,1)$. By strong monotonicity, $(0,\bar{\omega})$ and $(\bar{\omega},0)$ are also Pareto optimal. Hence every allocation on the diagonal is Pareto optimal. By our previous result (the existence of the one-to-one, continuous mapping $\varphi(\cdot)$), the diagonal exhausts all Pareto optimal allocations. Hence the Pareto set is equal to the diagonal.

15.B.8 Suppose that the preference of consumer i (i = 1,2) is represented by a utility function u_i: $\mathbb{R}^2_+ \to \mathbb{R}$ of the quasi-linear form $u_i(x_i) = x_{1i} + \phi_i(x_{2i})$, where ϕ_i: $\mathbb{R}_+ \to \mathbb{R}$ is continuous. We shall first prove that $\phi_i(\cdot)$ is strictly monotone and strictly concave. In fact, its strict monotonicity is an immediate consequence of that of the preference. As for the strict concavity, let $x_{2i} \neq x'_{2i}$ and $\lambda \in (0,1)$. By definition,

$$u_i(\phi_i(x'_{2i}) - \phi_i(0),\ x_{2i}) = u_i(\phi_i(x_{2i}) - \phi_i(0),\ x'_{2i}).$$

Hence, by the strict convexity of the preference,

$$u_i(\lambda(\phi_i(x'_{2i}) - \phi_i(0)) + (1 - \lambda)(\phi_i(x_{2i}) - \phi_i(0)),\ \lambda x_{2i} + (1 - \lambda)x'_{2i})$$
$$> \lambda u_i(\phi_i(x'_{2i}) - \phi_i(0),\ x_{2i}) + (1 - \lambda)u_i(\phi_i(x_{2i}) - \phi_i(0),\ x'_{2i}).$$

This is equivalent to

$$\phi_i(\lambda x_{2i} + (1 - \lambda)x'_{2i}) > \lambda\phi_i(x_{2i}) + (1 - \lambda)\phi_i(x'_{2i}).$$

Thus $\phi_i(\cdot)$ is strictly concave.

Now define φ: $[0,\bar{\omega}_2] \to \mathbb{R}$ by $\varphi(x_{21}) = \phi_1(x_{21}) + \phi_2(\bar{\omega}_2 - x_{21})$, then $\varphi(\cdot)$ is continuous and strictly concave. Hence there exists a unique maximum $x^*_{21} \in [0,\bar{\omega}_2]$.

In order to verify the assertion of the exercise, it is now sufficient to prove that if $x = (x_1,x_2) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++}$ is a nonwasteful feasible allocation in the interior of the Edgeworth box and $x_{21} \neq x^*_{21}$, then x is not Pareto optimal. In fact, let x be as such. Then, by the strict concavity of $\varphi(\cdot)$, for every $\lambda \in (0,1)$, we have

$$\varphi((1 - \lambda)x_{21} + \lambda x^*_{21}) > (1 - \lambda)\varphi(x_{21}) + \lambda\varphi(x^*_{21}) > \varphi(x_{21}).$$

For each i = 1,2, define $\delta_{\lambda i} \in \mathbb{R}$ as

$$-\phi_i((1-\lambda)x_{2i} + \lambda x^*_{2i}) + \phi_i(x_{2i}) + (1/2)(\varphi((1-\lambda)x_{21} + \lambda x^*_{21}) - \varphi(x_{21})),$$

then $\delta_{\lambda 1} + \delta_{\lambda 2} = 0$ by the definition of $\varphi(\cdot)$, and $\delta_{\lambda i} \to 0$ as $\lambda \to 0$ by continuity. Since $x_{1i} > 0$, we have $x_{1i} + \delta_{\lambda 1} > 0$ for each i for any sufficiently small $\lambda \in (0,1)$. Moreover,

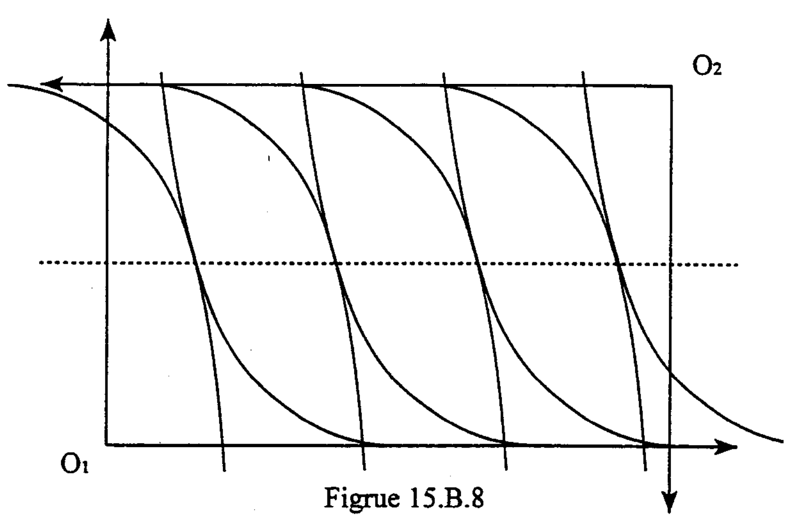
$$u_i(x_{1i} + \delta_{\lambda i}, (1-\lambda)x_{2i} + \lambda x^*_{2i}) = x_{1i} + \delta_{\lambda i} + \phi_i((1-\lambda)x_{2i} + \lambda x^*_{2i})$$

$$= x_{1i} + \phi_i(x_{2i}) + (1/2)(\varphi((1-\lambda)x_{21} + \lambda x^*_{21}) - \varphi(x_{21}))$$

$$> x_{1i} + \phi_i(x_{2i}) = u_i(x_{1i}, x_{2i}).$$

Thus x is not a Pareto optimal allocation. The figure below depicts the set of the Pareto optimal allocations in the interior of the Edgeworth box.

[It is worthwhile to note that if the consumptions sets were taken to be $(-\infty, \infty) \times \mathbb{R}$ as in Definition 3.B.7, then the Edgeworth "box" would have an infinite length in the horizontal direction and the assertion of this exercise could more easily be proved, without the interiority assumption or an explicit use of the utility functions. The reason is that, then, for any allocation x with $x_{21} \neq x^*_{21}$, there would always exist a feasible allocation x' with $x'_{21} = x^*_{21}$ that is Pareto superior to x. This is not always true when we have the nonnegativity constraints.]



Figrue 15.B.8

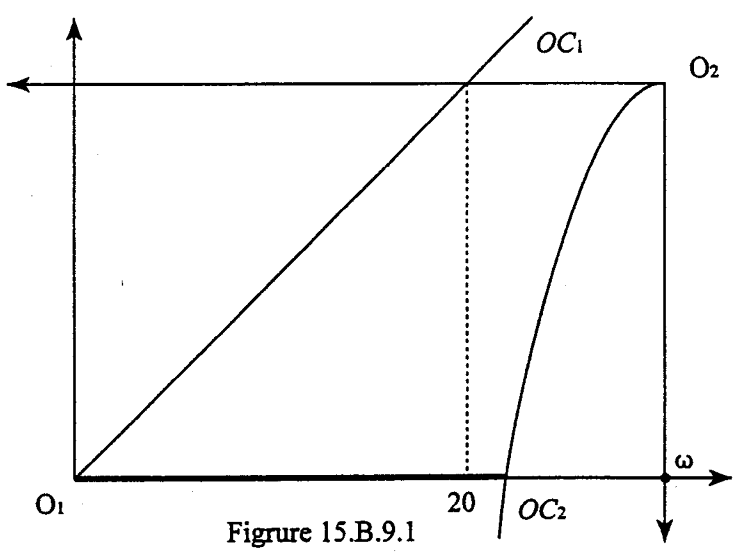
In connection with the discussion of Chapter 10, this fact implies that in the absence of wealth effects, a Pareto optimal allocation (in the interior of the Edgeworth box) of the non-numeraire commodities are uniquely determined. The differences in the consumers' utility levels at different Pareto optimal allocations (this time including the numeraire) can all be generated simply by redistributing the numeraire.

15.B.9 The offer curves of the two consumers are rather trivial when the prices of the two commodities are both positive. Their graphs are $\{(x_{p\alpha}, x_{b\alpha}) \in \mathbb{R}^2_{++}: x_{p\alpha} = x_{b\alpha}\}$ and $\{(x_{p\beta}, x_{b\beta}) \in \mathbb{R}^2_{++}: x_{p\beta} = (x_{b\beta})^{1/2}\}$. When one of the two commodities has zero price, the offer curves are given by

$$OC_\alpha(p_p,p_b) = \begin{cases} \{(x_{p\alpha},0): x_{p\alpha} \geq 0\} & \text{if } p_p = 0 \text{ and } p_b > 0, \\ \{(\omega_{p\alpha},x_{b\alpha}): x_{b\alpha} \geq \omega_{p\alpha}\} & \text{if } p_p > 0 \text{ and } p_b = 0. \end{cases}$$

$$OC_\beta(p_p,p_b) = \begin{cases} \{(x_{p\beta},20): x_{p\beta} \geq 20^{1/2}\} & \text{if } p_p = 0 \text{ and } p_b > 0, \\ \{(0,x_{b\alpha}): x_{b\alpha} \geq 0\} & \text{if } p_p > 0 \text{ and } p_b = 0. \end{cases}$$

These are depicted in the following figures. (Note that $4 < 20^{1/2} < 5$.)



Figrure 15.B.9.1

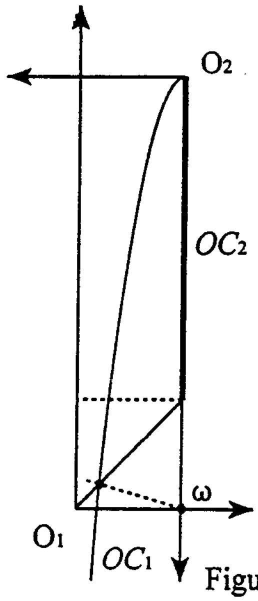


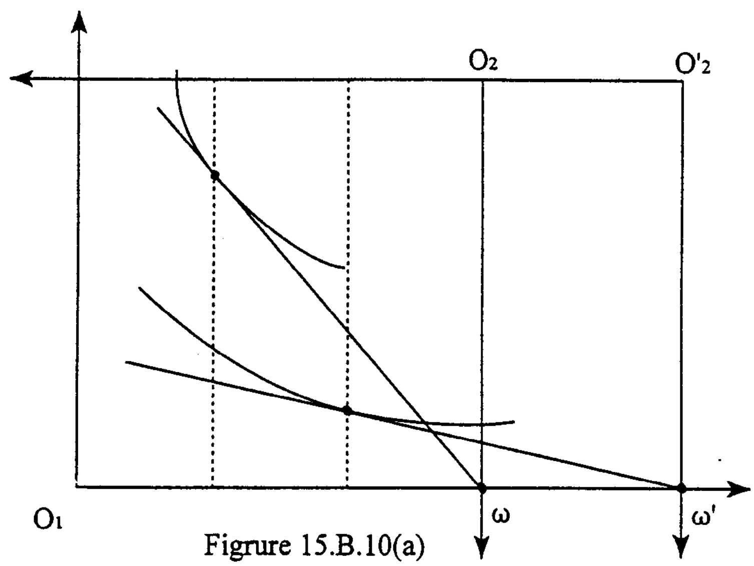
Figure 15.B.9.2

Thus, when $\omega_{p\alpha} = 30$, all equilibria are on the boundary of the Edgeworth box and the equilibrium price ratio and allocations are given by $p_p/p_b = 0$ and $\{((30 - x_{p\beta},0), (x_{p\beta}, 20)): 20^{1/2} \le x_{p\beta} \le 30\}$. When $\omega_{p\alpha} = 5$, the interior equilibrium is the intersection on the above figure . The boundary equilibria have the price ratio $p_b/p_p = 0$ and allocations $\{((5,x_{b\alpha}), (0, 20 - x_{b\alpha})): 5 \le x_{b\alpha} \le 20\}$.

Note that, when $\omega_{p\alpha}$ decreases from 30 to 5, Alphanse's utility level increases and Betatrix's utility level decreases, regardless of the choice of equilibria to be compared. This is because, when $\omega_{p\alpha} = 30$, Perrier is too abundant relative to Brie and its equilibrium price is zero, implying that Betatrix essentially consumes the total endowment of the economy. When $\omega_{p\alpha} =$ 5, Perrier is scarce enough to have positive price and Alphanse can afford positive consumptions of both goods. The price of Brie can even be driven down to zero, in which case he essentially consumes the total endowment of the economy.

15.B.10 **(a)** Suppose that the preferences of the two consumers are quasilinear with respect to commodity 2 and that $\omega_{21} = 0$. Suppose for a moment that $\omega'_{21} =$ 0, so that there is no increase in the endowment of consumer 1 for commodity

2. Here is an example in which an increase in the endowment of consumer 1 for commodity 1 may lead a decrease in his utility.



Figrure 15.B.10(a)

Since equilibrium allocations depend continuously on the initial endowments, we can still have a decrease in the utility of of consumer 1 when ω'_{21} is positive, but sufficiently small. We can thus take $\omega'_2 >> \omega_2$ as asked for in the exercise.

In this example, the small increment in the initial endowment leads to a substantial decrease in the relative price of commodity 1 . Since the wealth of consumer 1 comes exclusively from commodity 1, his real wealth then decreases, despite the increase in his endowment. Hence his utility decreases. This fact is often discussed in the theory of a quantity-setting monopoly: It is not in the monopolist's best interest to supply all it could potentially do, because an increase in supply leads a decrease in price.

(b) Let (p,x) be an equilibrium of the original endowments (ω_1,ω_2) and (p',x') be an equilibrium of the new endowments (ω'_1,ω'_2). In order to apply the result of Exercise 15.B.8, assume that both x and x' belong to the interior of the Edgeworth box. By the first fundamental theorem of welfare economics, both x and x' are Pareto optimal. Thus by the result of Exercise 15.B.8, we have x_{11}

$= x'_{11}$ and $x'_{12} = x'_{12}$. Hence, by the definition of quasilinearity (Definition 3.B.7), we can assume without loss of generality that p = p'. By the strong monotonicity of the preferences, p >> 0. Hence, by $\omega'_1 \geq \omega_1$ and $\omega'_1 \neq \omega_1$, we have $p \cdot \omega'_1 > p \cdot \omega_1$. Since $p \cdot x'_1 = p \cdot \omega'_1 > p \cdot \omega_1 = p \cdot x_1$, we obtain $x'_1 \succ_1 x_1$.

(c) Here is an example of the transfer paradox:

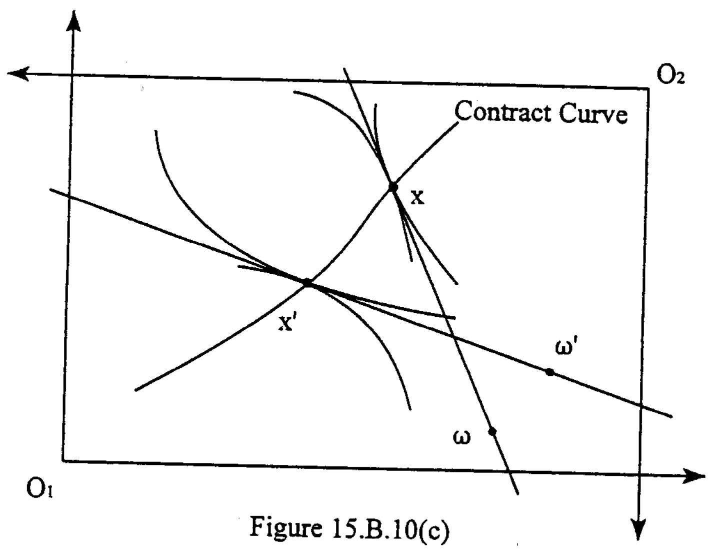


Figure 15.B.10(c)

In this example, a positive transfer is made from consumer 2 to consumer 1. If the price ratio were kept to be at the original level p_1/p_2, then there would be an excess demand for commodity 1. Thus the price ratio needs to change to recover an equilibrium. In this example, p_1/p_2 decreases, but this decrease induces a negative wealth effect on consumer 1 because he is the net supplier of commodity 1. Hence his equilibrium consumptions goes down from x_1 to x'_1.

(d) Following the hint, we shall prove that there are other equilibria at the original endowment ω. In the figure of (c), draw the budget line that goes through the original endowment ω and the new equilibrium x'. This budget line must be steeper than that of p' because $\omega_1 \ll \omega'_1$. Hence the demand of consumer 1 on this budget line must be in the north-west of x'. Of course, his offer curve with ω_1 must go through this demand, as well as the original equilibrium allocation x. Hence, as the relative price of commodity 1 increases, his offer curve must cut the contract curve from above at x. Symmetrically, as the relative price of commodity 1 increases, the offer curve of consumer 2 must cut the contract curve from below at x. This is illustrated in the following figure:

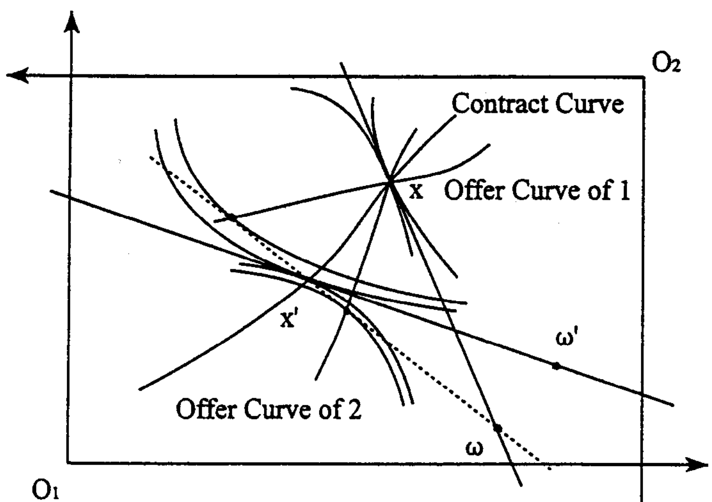


Figure 15.B.10(d)

Now recall that these two offer curves must go through the initial endowments. It will then be easy to convince yourself that, in whatever way you will extend the offer curves, they must intersect at at least two other points. Hence there are at least three equilibria with the original endowments.

15.C.1 **(a)** This is a simple consequence of two simple facts, both of which are already mentioned in the text: First, a Walrasian equilibrium is Pareto optimal. Second, under the strict convexity assumptions, there is a unique Pareto optimal allocation.

(b) Here is an example in which the slope of the excess demand function may change its sign. Here, given p = 1, the equilibrium wage level is w^* and, at wage levels w_1 and w_2 with $w_1 < w_2 < w^*$, we have $0 < z_1(w_1) < z_1(w_2)$. Hence

$z_1(\cdot)$ must have a positive slope somewhere between w_1 and w_2, and a negative slope somewhere between w_2 and w^*.

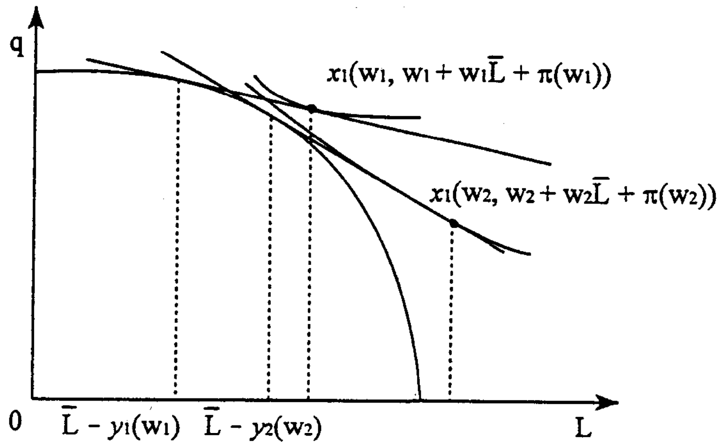


Figure 15.C.1(b)

To prove that the slope of the excess demand function is negative in a neighborhood of the equilibrium wage level w^*, assume its differentiability and denote the wealth level by $v = w\bar{L} + \pi(w)$, then

$$z_1'(w^*) = (\partial x_1/\partial w)(w^*, w^*\bar{L} + \pi(w^*)) + (\partial x_1/\partial v)(w^*, w^*\bar{L} + \pi(w^*))(\bar{L} + \pi'(w^*)) + y_1'(w^*).$$

Here, by (vi) of Proposition 5.C.1, $\pi'(w^*) = -\, y_1(w^*)$ and hence $\bar{L} + \pi'(w^*) = \bar{L} - y_1(w^*) = x_1(w^*, w^*\bar{L} + \pi(w^*))$. Thus the sum of the first two terms is equal to the diagonal element corresponding to labor of the Slutsky matrix of this consumer. Hence it is negative by Propositions 3.G.2 and 3.G.3. By (vii) of Proposition 5.C.1, $y_1'(w^*) < 0$. Hence $z_1'(w^*) < 0$.

(c) Suppose that the two consumers are endowed with the same amount of labor. Then, at any wage level, the total wealth of the economy is split equally between then. Here is an example of multiple equilibria. It is a modification of Example 4.C.1.

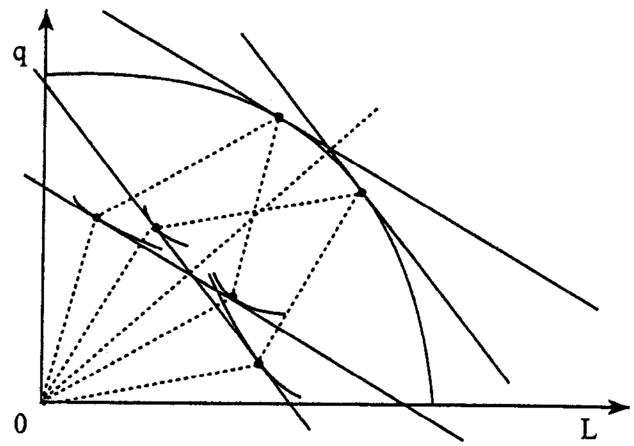


Figure 15.C.1(c.1)

If the firm operates under constant returns to scale, then there is a unique equilibrium allocation. To prove this, note first that the profit of the firm must be zero at any equilibrium. Thus, to find an equilibrium, we can assume that $\pi(w) = 0$ for every w. Since, in addition, the individuals are endowed with labor alone, they are always net suppliers of labor. Thus their offer curves look as follows.

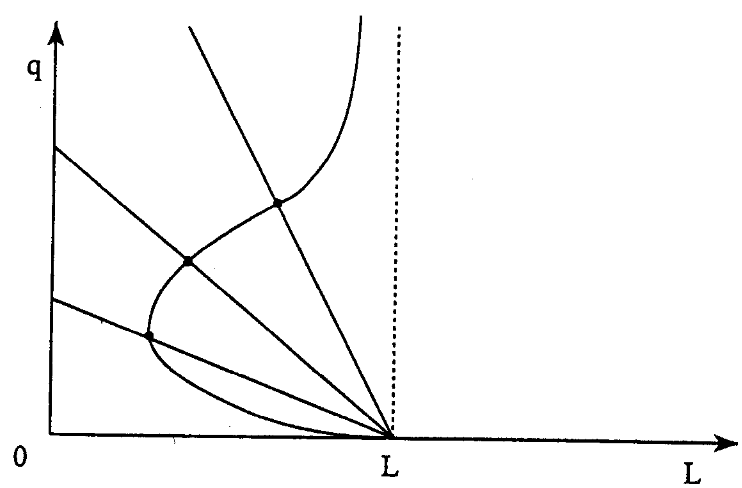


Figure 15.C.1(c.2)

Hence the total offer curve, that is, the sum of the two offer curves, also looks as above, and the equilibrium is described as an intersection of the total offer curves and the boundary of the production set, according to the profit maximization requirement of an equilibrium. Since the total offer curve can cross the boundary only once (otherwise, the labor demand function would be multi-valued), an equilibrium must be unique.

15.C.2 It is sufficient to calculate the Pareto optimal production levels, which is a solution to the maximization problem

$$u(z^{1/2}, 1 - z) = \ln z^{1/2} + \ln (1 - z).$$

By the first-order condition, $z = 1/3$. If we fix the output price to be one, then the equilibrium wage is $3^{1/2}/2$. The equilibrium profit is $1/(2\cdot 3^{1/2})$. The equilibrium consumption is $(1/3^{1/2}, 2/3)$.

15.D.1 **(a)** For any allocation $z = (z_1, z_2)$ in the Edgeworth box, we have $z_{21}/z_{11} > \bar{z}_2/\bar{z}_1 > z_{22}/z_{12}$ if and only if z lies above the diagonal; $z_{21}/z_{11} < \bar{z}_2/\bar{z}_1 < z_{22}/z_{12}$ if and only if z lies below the diagonal. Hence the assertion follows.

(b) For this question, first recall that the differentiability of the cost function of $c_j(\cdot)$ (or equivalently, the uniqueness of $a_j(w)$ at every w) is equivalent to saying that $f_j(\cdot)$ is strictly quasiconcave. This implies that the marginal rates of substitution changes strictly monotonically along the unit isoquant curve, and thus (together with the homogeneity of degree zero) that, for any z_j and z'_j, if $z_{2j}/z_{1j} \neq z'_{2j}/z'_{1j}$, then the marginal rates of substitution at z_j and z'_j are different.

Now suppose that a ray from the origin of firm 1 and the Pareto set of factor allocations intersect at $z = (z_1, z_2)$ (which is not the origin) and let $z' = (z'_1, z'_2)$ be another point on the ray. (The case in which a ray starts from the origin of firm 2 can be similarly proved.) It is sufficient to prove that z' is not Pareto optimal. By definition, $z_{21}/z_{11} = z'_{21}/z'_{11} \neq \bar{z}_2/\bar{z}_1$.

Thus $z_{22}/z_{12} \neq z'_{22}/z'_{12}$. The equality $z_{21}/z_{11} = z'_{21}/z'_{11}$ implies that the marginal rates of substitution of firm 1 at z_1 and z'_1 are the same. The inequality $z_{22}/z_{12} \neq z'_{22}/z'_{12}$ implies that the marginal rates of substitution of firm 2 at z_2 and z'_2 are different. By Pareto optimality, the marginal rate of substitution of firm 1 at z_1 and that of firm 2 at z_2 are the same. Therefore, the marginal rate of substitution of firm 1 at z'_1 and that of firm 2 at z'_2 are different. Hence z' is not Pareto optimal. Note that this result is equivalent to saying that the factor intensities at different Pareto optimal allocations are different.

Let's now show the (strict) monotonicity of the factor intensities and of the supporting relative factor prices along the Pareto set. It is sufficient to prove that of the former. By Exercise 15.B.7, there exist a bounded, closed (non-degenerated) interval $[\delta_0,\delta_1]$ and a continuous, one-to-one map $\varphi(\cdot)$ from $[\delta_0,\delta_1]$ into the Edgeworth box whose image coincides with the Pareto set of factor allocations. Define a map $\psi(\cdot)$ from $[\delta_0,\delta_1]$ into $\mathbb{R}$ by letting $\psi(\delta)$ be the factor intensity at $\delta \in [\delta_0,\delta_1]$, then $\psi(\cdot)$ is continuous. We want to prove that it is (strictly) monotone. In fact, if not, then the intermediate value theorem would imply that it is not one-to-one. That is, two different Pareto optimal allocations would have the same factor intensity. But this contradicts the result in the preceding paragraph. Hence $\psi(\cdot)$ is strictly monotone, implying the (strict) monotonicity of the factor intensity along the Pareto set.

15.D.2 Let $z = (z_1,z_2)$ and $z' = (z'_1,z'_2)$ be two feasible factor allocations. Let $\lambda \in [0,1]$. We want to prove that the consumption vector

$$\lambda(f_1(z_1),f_2(z_2)) + (1 - \lambda)(f_1(z'_1),f_2(z'_2))$$
$$= (\lambda f_1(z_1) + (1 - \lambda)f_1(z'_1),\ \lambda f_2(z_2) + (1 - \lambda)f_2(z'_2))$$

is in the production possibility set. Clearly,

$$(\lambda z_1 + (1 - \lambda)z_1') + (\lambda z_2 + (1 - \lambda)z_2')) \leq \bar{\omega}$$

and, by concavity,

$$f_1(\lambda z_1 + (1 - \lambda)z_1') \geq \lambda f_1(z_1) + (1 - \lambda)f_1(z_1'),$$

$$f_2(\lambda z_2 + (1 - \lambda)z_2') \geq \lambda f_2(z_2) + (1 - \lambda)f_2(z_2').$$

Hence the proof is completed.

15.D.3 We shall give two proofs. The first one uses Figure 15.D.6(a). The second one is more formal along the line of the Proof of Proposition 15.D.1. In both proofs, we consider the case in which the price of good 1 increases. The case in which the price of good 2 increases can be treated similarly.

For the first proof, suppose that the price of good 1 increases from p_1 to λp_1, where $\lambda > 1$. Let $w^* = (w_1^*, w_2^*)$ be the equilibrium factor price vector of (p_1, p_2) and $w^{**} = (w_1^{**}, w_2^{**})$ be that of $(\lambda p_1, p_2)$. We want to show that $w_1^{**} > \lambda w_1^*$. Of course, both w^* and λw^* are on the same ray. Since p_2 does not change, both w^* and w^{**} are on the same unit-cost curve $\{w: c_2(w) = p_2\}$ of firm 2, which is downward-sloping. By the equilibrium condition and the homogeneous of degree one of $c_1(\cdot)$, both w^{**} and λw^* are on the same unit-cost curve $\{w: c_1(w) = \lambda p_1\}$. The positions of w^*, λw^*, and w^{**} are depicted in the figure below:

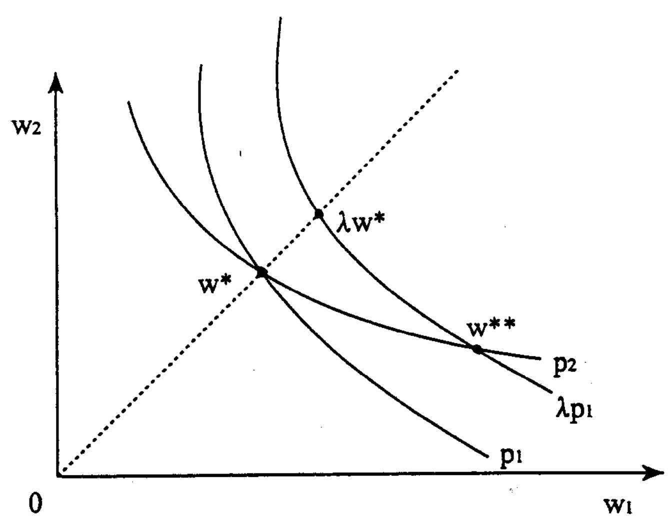


Figure 15.D.3

Hence we must have $w_1^{**} > \lambda w_1^*$.

For the second proof, note from the Proof of Proposition 15.D.1 that $dw_1 = (a_{22}(w^*)/|A|)dp_1$. Thus, by $p_1 = a_{11}(w^*)w_1^* + a_{21}(w_2^*)w_2^*$,

$$\begin{aligned} & dw_1/w_1^* - dp_1/p_1 \\ &= ((a_{22}(w^*)/|A|w_1^* - 1/p_1)dp_1 \\ &= \frac{a_{22}(w^*)}{w_1^*}\left(\frac{1}{|A|} - \frac{1}{a_{11}(w^*)a_{22}(w^*) + a_{21}(w^*)a_{22}(w^*)(w_2^*/w_1^*)}\right)dp_1 > 0. \end{aligned}$$

15.D.4 **(a)** The utility function of consumer i (i = 1,2) is denoted by $u_i(\cdot)$. The production function of firm j (j = 1,2) is denoted by $f_j(\cdot)$, which is assumed to be homogeneous of degree one. A price vector of the two consumption goods is denoted by $p = (p_1,p_2) \in \mathbb{R}^2_{++}$, a price vector of the two inputs by $w = (w_1,w_2) \in \mathbb{R}^2_{++}$, the consumption vector of consumer i by $x_i \in \mathbb{R}^2_+$, and the input demand of firm j by $z_j \in \mathbb{R}^2_+$. Write $x = (x_1,x_2)$ and $z = (z_1,z_2)$ for short. Then an equilibrium is defined as a vector (p^*,w^*,x^*,z^*) such that

1. (Utility Maximization) For each i, x_i^* solves the constraint maximization problem

$$\text{Max } u_i(x_i) \text{ s.t. } p^*\cdot x_i \le w_i^*.$$

2. (Profit Maximization) For each j, z_j^* solves the constraint maximization problem

$$\text{Max } p_j^* f_j(z_j) - w^*\cdot z_j \text{ s.t. } z_j \in \mathbb{R}^2_+.$$

3. (Market Clearing) $\sum_i x_i^* = (f_1(z_1^*),f_2(z_2^*))$ and $\sum_j z_j^* = (1,1)$.

Of course, Condition 2 can be replaced by the first-order condition $p_j^* = c_j(w_1^*,w_2^*)$ and $z_{1j}^*/z_{2j}^* = a_{1j}(w^*)/a_{2j}(w^*)$.

(b) Suppose now that we have two equilibria (p^*,w^*,x^*,z^*) and $(p^{**},w^{**},x^{**},z^{**})$. Assume without loss of generality that $p_1^* = p_1^{**} = 1$ and $f_1(z_1^*) \le f_1(z_1^{**})$ and $f_2(z_2^*) \ge f_2(z_2^{**})$. Note that $w_1^*/p_2^* = f_2(z_2^*)$, $w_2^* = f_1(z_1^*)$, $w_1^{**}/p_2^{**} = f_2(z_2^{**})$, $w_2^{**} = f_1(z_1^{**})$ by the utility maximization and the market clearing. According to Exercise 15.D.1(b), the input allocations z^* and z^{**} can be depicted as follows:

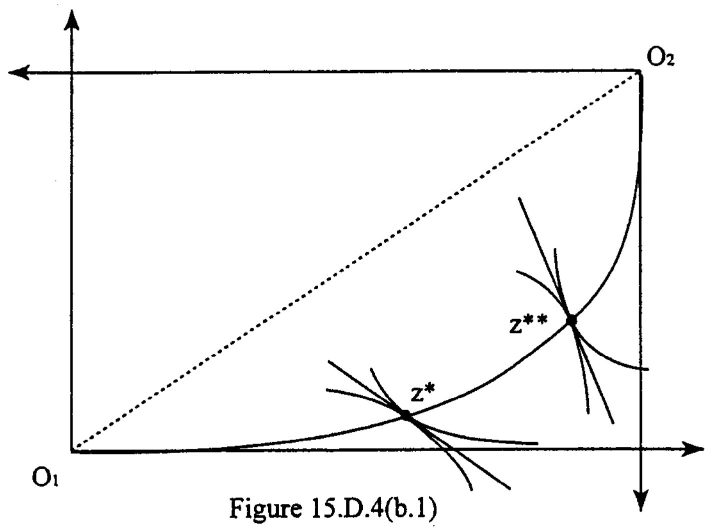


Figure 15.D.4(b.1)

Hence $w_2^*/w_1^* \geq w_2^{**}/w_1^{**}$. Thus, as we can see with the unit cost curves (and because of $p_1^* = p_1^{**} = 1$), we must have $w_2^* \geq w_2^{**}$, that is, the price of input 2 cannot increase even in the absolute value:

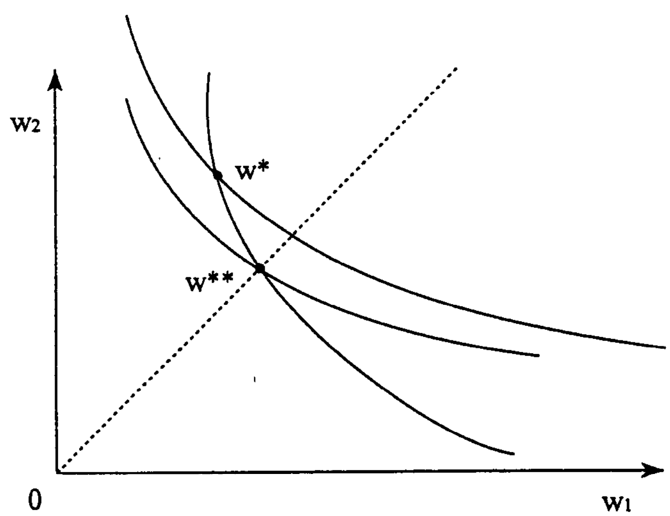


Figure 15.D.4(b.2)

Hence $f_1(z_1^*) \geq f_1(z_1^{**})$. Thus $f_1(z_1^*) = f_1(z_1^{**})$ and $f_2(z_2^*) = f_2(z_2^{**})$. Since the $f_j(\cdot)$ are strictly quasiconcave, this implies that $z^* = z^{**}$. Hence, by $p_1^* = p_1^{**} = 1$, $w^* = w^{**}$ and $p_2^* = p_2^{**}$.

(c) Suppose now that we have two equilibria (p^*, w^*, x^*, z^*) and $(p^{**}, w^{**}, x^{**}, z^{**})$. Assume without loss of generality that $p_1^* = p_1^{**} = 1$. We only show that it is not inconsistent to have $f_1(z_1^*) < f_1(z_1^{**})$, $f_2(z_2^*) > f_2(z_2^{**})$, $w_1^* = f_1(z_1^*)$, $w_1^{**} = f_1(z_1^{**})$, $w_1^*/p_2^* = f_2(z_2^*)$, and $w_2^{**}/p_2^{**} = f_2(z_2^{**})$ at the same time. In fact, again, we know from the unit cost curves that $w_1^* < w_1^{**}$ and $w_2^* > w_2^{**}$. But, as we saw in the proof of Exercise 15.D.3, the profit maximization and the factor market clearing implies that $w_2^{**}/w_2^* < p_2^{**}/p_2^*$, that is, $w_2^{**}/p_2^{**} < w_2^*/p_2^*$. This last inequality is nothing but $f_2(z_2^*) > f_2(z_2^{**})$.

15.D.5 Denote the initial factor allocation by $z = (z_1, z_2)$ and the new factor allocation by $z' = (z_1', z_2')$, after the endowment of input 1 increases from $\bar{z}_1$ to $\bar{z}_1'$. Note on Figure 15.D.7 that z_j and z_j' are proportional for each $j = 1,2$, and that $z_1' >> z_1$. In particular, $z_{21}' > z_{21}$. Since the endowment of input is fixed at the level of $\bar{\omega}_2$, this implies that $z_{22}' < z_{22}$. By proportionality, $z_{12}' < z_{12}$. Thus, $\bar{z}_1' - z_{11}' < \bar{z}_1 - z_{11}$, that is, $z_{11}' - z_{11} > \bar{z}_1' - \bar{z}_1$. Hence, by dividing the left hand side by z_{11} and the right hand side by $\bar{z}_1$ (and because of $z_{11} < \bar{z}_1$), we obtain $z_{11}'/z_{11} > \bar{z}_1'/\bar{z}_1$. By the homogeneity of degree one and the proportionality, we have $f_1(z')/f_1(z) > \bar{z}_1'/\bar{z}_1$.

15.D.6 **(a)** Writing $w^* = (w_1^*, w_2^*)$, the equilibrium conditions for w^* and (q_1^*, q_2^*) are that

$$c_1(w^*) \geq p_1, \text{ and } c_1(w^*) = p_1 \text{ whenever } q_1^* > 0;$$

$$c_2(w^*) \geq p_2, \text{ and } c_2(w^*) = p_2 \text{ whenever } q_2^* > 0;$$

$$a_{11}(w^*)q_1^* + a_{12}(w^*)q_2^* = \bar{z}_1;$$

$$a_{21}(w^*)q_1^* + a_{22}(w^*)q_2^* = \bar{z}_2.$$

(b) Suppose first that an equilibrium $\hat{w}$ and (q_1^*, q_2^*) satisfies

$$c_1(\hat{w}) = p_1 \text{ and } q_1^* > 0;\ c_2(\hat{w}) = p_2 \text{ and } q_2^* > 0.$$

By the factor intensity condition, (the inverse of) the slope $a_{11}(\hat{w})/a_{21}(\hat{w})$

of the optimal input vector of firm 1 must be greater the slope $a_{12}(\hat{w})/a_{22}(\hat{w})$ of the optimal input vector of firm 2. By the market clearing condition and $q_1^* > 0$, $q_2^* > 0$, the slope $\bar{z}_1/\bar{z}_2$ of the endowment vector must be between these two slopes. This is equivalent to saying that $\bar{z}$ belongs to the diversification cone.

Suppose conversely that $\bar{z}$ belongs to the diversification cone. If $q_1^* = 0$, then the market clearing condition implies that $q_2^*(a_{12}(\hat{w}),a_{22}(\hat{w})) = \bar{z}$. Thus $a_{12}(\hat{w})/a_{22}(\hat{w}) = \bar{z}_1/\bar{z}_2$ and hence $\bar{z}$ does not belong to the diversification cone. Similarly, if $q_2^* = 0$, then z does not belong to the diversification cone either. We must thus have $(q_1^*,q_2^*) \gg 0$.

(c) We shall prove the assertion of this question by showing that if the unit-dollar isoquants intersect more than once, then the factor intensity condition is not satisfied. The next paragraph is devoted to a proof of the statement in the hint that if they intersect more than once, then there are two points (one in each isoquant) proportional to each other and such that the slopes of the isoquants at these points are identical. As the proof is technical (and perhaps unnecessarily long), it can be skipped. The slope of the unit-dollar isoquant of firm j at point (z_{1j},z_{2j}) is denoted by $s_j(z_{1j})$. (Since, for each z_{1j}, there is only one z_{2j} such that (z_{1j},z_{2j}) lies on the isoquant, we can suppress z_{2j} from the augment of the slope function $s_j(\cdot)$.)

Let $v \in \mathbb{R}^2_{++}$ and $v' \in \mathbb{R}^2_{++}$ be two different intersections of the two isoquants. Denote by C the region of the unit-dollar isoquant of firm 1 bounded by v and v' inclusive, then C is a compact set. If there are infinitely many intersections of the two isoquants on C, let $\{v^n\}$ be a sequence of different intersections in C. We can assume without loss of generality that it converges to a point $\bar{v} \in C$. Then $\bar{v}$ is also an intersection. Since $s_j(\bar{v}_1) = \lim \dfrac{v_2^n - \bar{v}_2}{v_1^n - \bar{v}_1}$, we have $s_1(\bar{v}_1) = s_2(\bar{v}_1)$. Hence the hint is verified for this case of infinitely many intersections on C. If there are only finitely many intersections on C, pick up two consecutive intersections. To simplify notation, let v and v' be as such and $v_1 < v'_1$. Since one of the two isoquants is above the other everywhere between v and v', we have $(s_1(v_1) - s_2(v_1))(s_1(v'_1) - s_2(v'_1)) \leq 0$. If we have either $s_1(v) = s_2(v)$ or $s_1(v') = s_2(v')$, then there is nothing to prove. So suppose not, then $(s_1(v_1) - s_2(v_1))(s_1(v'_1) - s_2(v'_1)) < 0$. Suppose also that the isoquant of firm 2 is above that of firm 1 everywhere between v and v'. (The other case can be proved similarly.) Then $s_1(v_1) < s_2(v_1) < 0$ and $s_2(v'_1) < s_1(v'_1) < 0$. For each $z_{11} \in [v_1,v'_1]$, let $\lambda(z_{11}) \geq 1$ be such that, if (z_{11},z_{21}) lies on the isoquant of firm 1, then $\lambda(z_{11})(z_{11},z_{21})$ lies on that of firm 2. Note that $\lambda(v_1) = \lambda(v'_1) = 1$ and $\lambda(z_{11})z_{11} \in [v_1,v'_1]$ for every $z_{11} \in [v_1,v'_1]$. Now define $g\colon [v_1,v'_1] \to \mathbb{R}$ by $g(z_{11}) = s_1(z_{11}) - s_2(\lambda(z_{11})z_{11})$, then $g(v_1) < 0$ and $g(v'_1) > 0$. By the continuity of $g(\cdot)$ (which is implied by the continuous differentiability of the $f_j(\cdot)$) and the intermediate value theorem, there must exist $z_{11} \in (v_1,v'_1)$ such that $g(z_{11}) = 0$. But this implies that $s_1(z_{11}) = s_2(\lambda(z_{11})z_{11})$, that is, if (z_{11},z_{21}) lies on the isoquant of firm 1, then $\lambda(z_{11})(z_{11},z_{21})$ lies on that of firm 2 and the slopes of the two isoquants are the same on those two points. The Hint is thus proved.

Now suppose that there are more than one intersections. By the hint, there are z_1^* and $\lambda > 0$ such that z_1^* lies on the unit-dollar isoquant of firm 1, λz_1^* lies on the unit-dollar isoquant of firm 2, and the slopes at those points are the same. Hence if $w = (w_1,w_2) \in \mathbb{R}^2_{++}$ is such that w_2/w_1 is equal to the slope, then z_1^* attains the minimum of $w \cdot z_1$ on $\{z_1\colon f_1(z_1) \geq 1/p_1\}$ and λz_1^* is the minimum of $w \cdot z_2$ on $\{z_2\colon f_2(z_2) \geq 1/p_2\}$. By homogeneity of degree one, this implies that $p_1 z_1^*$ attains the minimum of $w \cdot z_1$ on $\{z_1\colon f_1(z_1) \geq 1\}$ and $p_2 \lambda z_1^*$ attains the minimum of $w \cdot z_2$ on $\{z_2\colon f_2(z_2) \geq 1\}$. But since $p_1 z_1^*$ and $p_2 \lambda z_1^*$ are proportional, the factor intensity condition is not satisfied.

As for the graphical construction of the diversification cone, if $\hat{w}$ is an equilibrium input price vector, then, for each j, $\hat{w}$ supports $\{z_j: f_j(z_j) \geq 1/p_j\}$ at $(1/p_j)(a_{1j}(\hat{w}),a_{2j}(\hat{w}))$. Moreover $\hat{w}\cdot((1/p_j)(a_{1j}(\hat{w}),a_{2j}(\hat{w}))) = 1$. Hence we obtain the following figure.

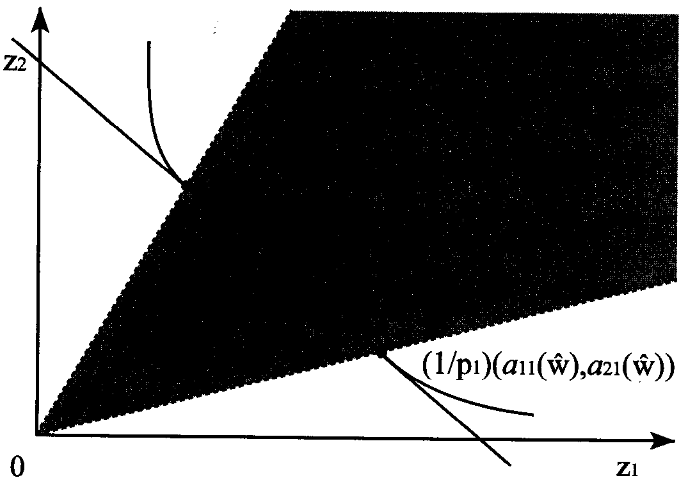


Figure 15.D.6(c)

(d) As we saw in the discussion preceding Proposition 15.D.2 (Rybcszynski Theorem), the factor intensity condition implies that there exists exactly one factor price vector $\hat{w} = (\hat{w}_1,\hat{w}_2)$ such that, for any total initial endowments, the factor price vector of any equilibrium involving positive production of both goods is equal to $\hat{w}$. By (b), the total initial endowment vector $\bar{z}$ gives rise to an (unique) equilibrium that involves positive production of both goods if and only if $\bar{z}$ belongs to the diversification cone of $\hat{w}$. If $\bar{z}$ lies below the cone, that is, $\bar{z}_1/\bar{z}_2 \geq a_{11}(\hat{w})/a_{21}(\hat{w})$, then the economy specializes in production of good 1 and the equilibrium factor price vector w* is determined so that $a_1(w^*) = (1/f_1(\bar{z}))\bar{z}$ and $c_1(w^*) = p_1$. If, on the other hand, $\bar{z}$ lies above the cone, that is, $\bar{z}_1/\bar{z}_2 \leq a_{12}(\hat{w})/a_{22}(\hat{w})$, then the economy specializes in production of good 2 and the equilibrium factor price vector w** is determined so that $a_2(w^{**}) = (1/f_2(\bar{z}))\bar{z}$ and $c_2(w^{**}) = p_2$. These are illustrated in the following picture.

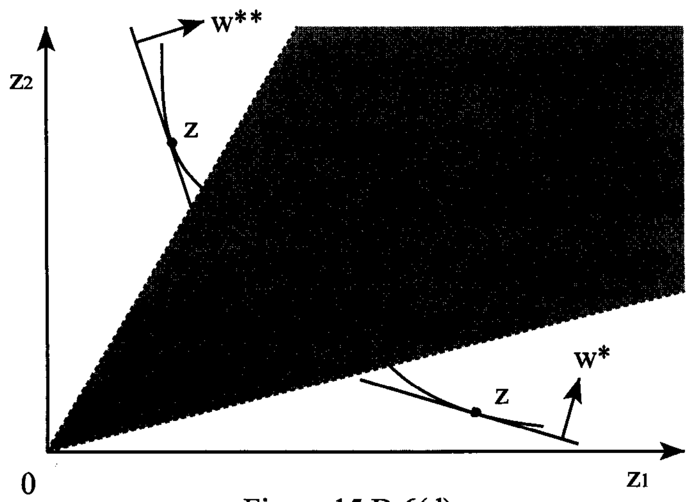


Figure 15.D.6(d)

15.D.7 It is straightforward either from (15.D.1) and (15.D.2) or from (15.D.5) that $z^*_{11} = 4\bar{z}_1/5$, $z^*_{21} = \bar{z}_2/5$, $z^*_{12} = \bar{z}_1/5$, $z^*_{22} = 4\bar{z}_2/5$ $w^*_1 = 5^{1/2}/2\bar{z}_1^{-1/2}$, and $w^*_2 = 5^{1/2}/2\bar{z}_2^{-1/2}$.

15.D.8 It is easy to check that the production of good 1 is relatively more intensive in factor 1 than in the production of good 2. We can thus apply the graphical apparatus obtained in Exercise 15.D.6 to answer this question. By some straightforward calculations, we can show that the unique equilibrium factor price vector $\hat{w} = (\hat{w}_1, \hat{w}_2)$ that involves positive production of both goods is equal to $(2^{2/3}/3, 2^{2/3}/3)$ and is independent of the total factor endowments; and that $a_{11}(\hat{w}) = 2^{1/3}$, $a_{21}(\hat{w}) = 2^{-2/3}$, $a_{12}(\hat{w}) = 2^{-2/3}$, $a_{22}(\hat{w}) = 2^{1/3}$. Since the unit-dollar isoquants are equal to the (standard) isoquants $\{z_j \in \mathbb{R}^2_+: f_j(z_j) = 1\}$ by p = (1,1), these results yield the following figure:

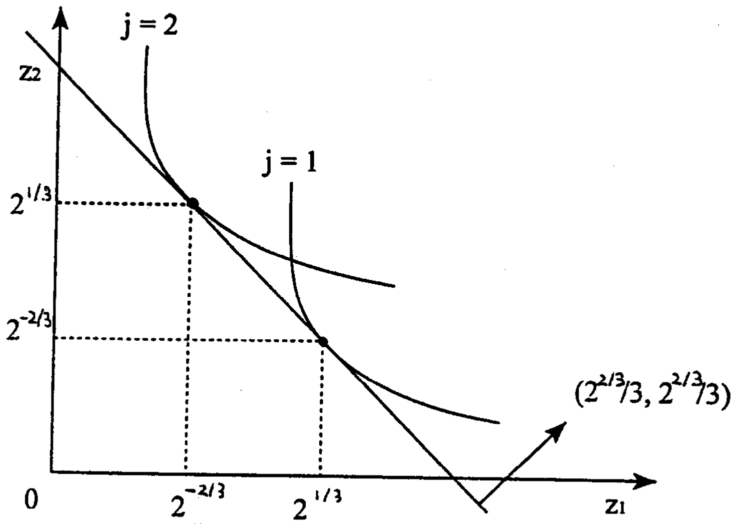


Figure 15.D.8

Hence the total factor endowments $\bar{z}$ gives rise to positive production of both goods if and only if $2^{-2/3}/2^{1/3} < \bar{z}_1/\bar{z}_2 < 2^{1/3}/2^{-2/3}$, that is, $1/2 < \bar{z}_1/\bar{z}_2 < 2$. The equilibrium production level $q^* = (q_1^*, q_2^*)$ must then satisfy $\bar{\omega} = A^T q^*$, where A is the 2×2 matrix as defined in the Proof of Proposition 15.D.1 and A^T is its transpose. Solving this, we obtain the equilibrium factor allocation:

$$z_{11}^* = q_1^* a_{11}(\hat{w}) = (2/3)(2\bar{z}_1 - \bar{z}_2),\quad z_{21}^* = q_1^* a_{21}(\hat{w}) = (1/3)(2\bar{z}_1 - \bar{z}_2),$$

$$z_{12}^* = q_1^* a_{12}(\hat{w}) = (1/3)(2\bar{z}_2 - \bar{z}_1),\quad z_{22}^* = q_2^* a_{22}(\hat{w}) = (2/3)(2\bar{z}_2 - \bar{z}_1).$$

If $\bar{z}$ lies below the diversification cone, that is, $\bar{z}_1/\bar{z}_2 \geq 2$, then the economy specializes in production of good 1. The equilibrium factor price vector w^* is determined so that $a_1(w^*) = (1/f_1(\bar{z}))\bar{z}$ and $c_1(w^*) = p_1$. Thus

$$w^* = \frac{1}{2^{-2/3}\bar{z}_1 + 2^{4/3}\bar{z}_2}(2\bar{z}_2, \bar{z}_1).$$

Symmetrically, if $\bar{z}_1/\bar{z}_2 \leq 1/2$, then the economy specializes in production of good 2. The equilibrium factor price vector w^* is given by

$$w^* = \frac{1}{2^{4/3}\bar{z}_1 + 2^{-2/3}\bar{z}_2}(\bar{z}_2, 2\bar{z}_1).$$

15.D.9 Let $(p^*, w_A^*, w_B^*, z_A^*, z_B^*, q_A^*, q_B^*, x_A^*, x_B^*)$ be an equilibrium of this two-country model, where $p^* \in \mathbb{R}^2_{++}$ is the international price vector of the two consumption goods, $w^*_A \in \mathbb{R}^2_{++}$ is the factor price vector in country A, $z^*_A \in \mathbb{R}^4_{++}$ is the factor allocation in country A, $q^*_A \in \mathbb{R}^2_{++}$ is the output levels of the two consumption goods in country A, and $x^*_A \in \mathbb{R}^2_{++}$ is the aggregate demand for the two consumption goods of country A, and similarly for country B. By the assumptions on the utility functions, x^*_A and x^*_B are proportional. By the market clearing condition, $x^*_A + x^*_B = q^*_A + q^*_B$ and this sum is proportional to x^*_A and x^*_B. By the budget constraints, $p^* \cdot (x^*_A - q^*_A) = p^* \cdot (x^*_B - q^*_B) = 0$. Hence, in order to verify this theorem, it is sufficient to prove that $q^*_{1A}/q^*_{2A} > q^*_{1B}/q^*_{2B}$.

We shall now prove this inequality. Let $\lambda = \bar{z}_{21}/\bar{z}_{22} > 0$ and consider an auxiliary country C, which is endowed with the total factor allocation $\lambda \bar{z}_2 \in \mathbb{R}^2_{++}$. It is easy to see that, faced with the international price vector p^* of the two consumption goods, the factor price vector w^*_B would clear the input markets in country C; the corresponding factor allocation would be equal to λz^*_B and the output levels to λq^*_B. Country A is endowed with the same amount of factor 2 as country C and a larger amount of factor 1 than country C. Hence (as neither of country A nor C specializes) the Rybcszynski Theorem is applicable to comparison between countries A and C, implying that $q^*_{1A} > \lambda q^*_{1B}$ and $q^*_{2A} < \lambda q^*_{2B}$. Thus $q^*_{1A}/q^*_{2A} > \lambda q^*_{1B}/\lambda q^*_{2B} = q^*_{1B}/q^*_{2B}$.